# AMERICAN
# Journal of Mathematics

EDITED BY

**FRANK MORLEY**

WITH THE COOPERATION OF

A. COHEN, CHARLOTTE A. SCOTT, A. B. COBLE

AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

*Πραγμάτων ἔλεγχος οὐ βλεπομένων*

VOLUME XLI

BALTIMORE: THE JOHNS HOPKINS PRESS

LEMCKE & BUECHNER, *New York.* E. STEIGER & CO., *New York.* WILLIAM WESLEY & SON, *London*
G. E. STECHERT & CO., *New York.* ARTHUR F. BIRD, *London.* A. HERMANN, *Paris.*

1919

The Lord Baltimore Press

BALTIMORE, MD., U. S. A.

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JANUARY, 1919

Entered as Second-Class Matter at the Baltimore, Maryland, Postoffice.
Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917.
Authorized on July 3, 1918.

# *Groups Generated by Two Operators Whose Relative Transforms are Equal to Each Other.*

By G. A. Miller.

If $s$ and $t$ represent two operators of a group the important operator

$$s^{-1}ts$$

is known as the *transform* of $t$ with respect to $s$. Similarly $t^{-1}st$ is called the transform of $s$ with respect to $t$. These two transforms are usually distinct. When $s$ and $t$ are commutative they are equal to each other whenever $s$ and $t$ are identical and only then. When $s$ and $t$ are non-commutative the equality of these transforms still implies, among other things, that $s$ and $t$ have the same order, since the order of an operator is equal to that of all its transforms. The main object of the present paper is to determine fundamental properties of the groups generated by $s$ and $t$ when they satisfy the equation

$$s^{-1}ts=t^{-1}st, \tag{A}$$

but are not otherwise restricted.

Our first object is to determine the order of $t^{-1}s$ when the common order of $s$ and $t$ is an arbitrary positive integer $\alpha$. For the determination of this order it is convenient to employ the equation

$$(t^{-1}s)^2=st^{-1}, \tag{B}$$

which can readily be established by writing (A) in the form

$$t^{-1}s=s^{-1}tst^{-1},$$

and substituting in the second member of

$$(t^{-1}s)^2=t^{-1}st^{-1}s.$$

From (B) it results directly that $t^{-1}s$ is transformed into its square by $s^{-1}$. Hence $s$ transforms into itself the cyclic group generated by $t^{-1}s$, and the group $G$ generated by this cyclic group and $s$ is identical with the group generated by $s$ and $t$. In particular, it results that $G$ is of finite order provided $t^{-1}s$ is of finite order. We shall now prove that the latter order is a divisor of

$$2^\alpha-1=2^{\alpha-1}+2^{\alpha-2}+\ldots.+2+1.$$

This fact can easily be established by means of (B). When $\alpha=1$ no proof is required. When $\alpha>1$, sets of four factors in the first member of the following equation, beginning at the left,

$$t^{-1}st^{-1}s\ldots.t^{-1}s=(t^{-1}s)^{2^\alpha-1}$$

can be replaced by sets of two factors by means of (B). We thus obtain the simpler equation

$$st^{-1}st^{-1}\ldots.t^{-2}s=(st^{-1})^{2^{\alpha-1}-1}\cdot t^{-1}s=s(t^{-1}s)^{2^{\alpha-1}-2}\cdot t^{-2}s.$$

When $\alpha=2$ it is evident that the members of the last equation reduce to identity. When $\alpha>2$ we may write out the last member of this equation and in it affect a similar reduction by means of (B), thus obtaining the equation

$$s^2t^{-1}st^{-1}s\ldots.t^{-3}s=s^2(t^{-1}s)^{2^{\alpha-2}-2}\cdot t^{-3}s.$$

The members of this equation are evidently equal to the identity when $\alpha=3$. When $\alpha>3$ the process may clearly be repeated until the identity is reached. Hence the theorem:

*If $s$ and $t$ are two operators of order $\alpha$ which satisfy the condition $s^{-1}ts=t^{-1}st$, then the order of $t^{-1}s$ is a divisor of $2^\alpha-1$.*

The fact that the order of $t^{-1}s$ must be exactly $2^\alpha-1$ whenever no additional restriction is placed on $s$ and $t$, may be established by actually constructing a group involving operators which satisfy these conditions. We proceed to do this. Let $t_1$ represent an operator of order $2^\alpha-1$ and let $s_1$ represent an operator of order $\alpha$ in the group of isomorphisms of the cyclic group generated by $t_1$. Since 2 belongs to exponent $\alpha$ mod $2^\alpha-1$ it may be assumed that $s_1t_1s_1^{-1}=t_1^2$, and hence

$$t_1s_1^{-1}t_1^{-1}=s_1^{-1}t_1.$$

The operators $s_1$ and $s_1t_1^{-1}$ are therefore of order $\alpha$. Their relative transforms are equal to each other since

$$s_1^{-1}\cdot s_1t_1^{-1}s_1=t_1^{-1}s_1 \text{ and } t_1s_1^{-1}\cdot s_1\cdot s_1t_1^{-1}=t_1s_1t_1^{-1}.$$

In fact, the second members of these two equations are the inverses of operators which were proved equal to each other and hence they must also be equal to each other.

From the preceding paragraph it results that there are two operators of order $\alpha$, $\alpha$ being any positive integer, whose relative transforms are equal to each other and which satisfy the condition that the product of one and the inverse of the other is of order $2^\alpha-1$. Hence the following theorem has been established:

*If the relative transforms of two operators of a group are equal to each other these operators have the same order $\alpha$ and when they are not otherwise restricted they generate a solvable group of order $\alpha$ $(2^\alpha-1)$.*

For a particular value of $\alpha$ there is one and only one such group. This group contains a cyclic commutator subgroup of order $2^\alpha-1$ and is generated by this subgroup and an operator of order $\alpha$ which transforms each of its operators into its square. In view of the great importance of the concept of transform in the theory of groups and the remarkable simplicity of these groups it may be desirable to denote this system by a special name. We shall call it the *equi-transform system* of groups.

For the sake of illustrations it may be desirable to note that when $\alpha=1$ this group is the identity, when $\alpha=2$ it is the dihedral group of order 6, when $\alpha=3$ it is the semi-metacyclic group of order 21, etc. It is evident that each of the groups in the equi-transform system contains a set of $2^\alpha-1$ conjugate cyclic subgroups of order $\alpha$ which are therefore separately transformed into themselves by only their own operators. Moreover, when $\alpha$ is even and represented by $2n$ all the $2^n+1$ operators of order 2 contained in such a group constitute a single set of conjugates. The fact that each of these operators of order 2 appears in at least one set of independent generators of $G$ results directly from the following evident theorem:

*If each one of a complete set of conjugate operators of a group can be transformed into each of the other operators of the set by some operator of the set then each of these operators appears in at least one set of independent generators of the group.*

From the preceding paragraph it results directly that the $\phi$-subgroup of $G$ must always be of odd order. We proceed to prove that the order of this subgroup is $2^\alpha-1$ divided by the product of all its distinct prime factors. To prove this theorem it is only necessary to prove that every operator of $G$ which is not found in its invariant cyclic subgroup of order $2^\alpha-1$ does appear in at least one of the sets of independent generators of $G$. If this theorem were not true there would be such an operator $S$ of prime order $p$ which would not belong to a set of conjugates having the properties noted in the theorem closing the preceding paragraph.

Since $S$ would be a power of an operator $s$ of order $\alpha$ contained in $G$ it may be assumed that $\alpha=kp$ and that $s^{-1}ts=t^2$, $t$ being a generator of the invariant cyclic subgroup of order $2^\alpha-1$ contained in $G$. Hence

$$S^{-1}tS=t^{2^k}.$$

It may be noted that when $2^a-1$ is prime to $p$ the theorem is evident. Hence we shall assume in what follows that $2^a-1$ is divisible by $p$. Moreover, it may be assumed that $S$ is commutative with the Sylow subgroups whose orders are prime to $p$ contained in the invariant cyclic subgroup of order $2^a-1$. Hence it results that

$$2^k-1\equiv 0 \mod (2^a-1)/p.$$

It is easy to prove that this congruence is impossible. In fact,

$$p(2^k-1)<2^a-1$$

since $p\cdot 2^k<2^a=2^{kp}$. Hence the operator $S$ belongs to a set of conjugates such that each of the operators of the set is transformed into every operator of the set by some operator of the set, and the following theorem has been established:

*The $\phi$-subgroup of every group in the equi-transform system is cyclic and of order $2^a-1$ divided by the product of the distinct prime factors of this number.* Incidentally it has also been proved that *every prime factor of $a$ is a divisor of the remainder obtained by subtracting unity from some prime factor of $2^a-1$.*

UNIVERSITY OF ILLINOIS.

# *A Classification of General (2, 3) Point Correspondences Between Two Planes.*

By Temple Rice Hollcroft.

## § 1. Introduction and General Discussion.

1. *Introduction.*—The purpose of this paper is to discuss and classify algebraic (2, 3) point correspondences between two planes. Such a correspondence is established between two planes $(x)$ and $(x')$ when an algebraic relation in $x_1, x_2, x_3$ and $x'_1, x'_2, x'_3$ exists such that to any point of $(x)$ shall correspond two points of $(x')$, and to any point of $(x')$ three points of $(x)$.

While the theory of (1, 1) correspondences has been thoroughly developed and that of $(1, n)$ correspondences treated in some detail, there are but few writings that consider the case in which both planes are multiple and thus have a direct bearing on the subject of the present paper. The first paper on (2, 2) point correspondences was published in 1889 by P. Visalli.* In this paper he discusses a very special case in which the lines of either plane correspond to conics in the other. Burali-Forti† later obtained certain (2, 2) correspondences by combining two (1, 2) correspondences and showed that the case treated by Visalli is included in these. The first investigation of general (2, 2) point correspondences was made by G. Marletta.‡ He considers two special types and deduces the properties of their associated transformations geometrically in four dimensional space. Finally an exhaustive classification of (2, 2) point correspondences has been recently made by F. R. Sharpe and Virgil Snyder.§

The only paper to date on correspondences of multiplicities greater than two is that of Richard Baldus,‖ who has investigated certain properties of $(m_1, m_2)$ point correspondences by geometrical methods.

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* P. Visalli, "La trasformazione quadratica (2, 2)," *Rend. del Circ. Mat. di Palermo,* Vol. III (1889), pp. 165–170.

† C. Burali-Forti, "Sulle trasformazione (2, 2) che si possono ottenere mediante due trasformazioni doppie," *Rend. del Circ. Mat. di Palermo,* Vol, V (1891), pp. 91–99.

‡ G. Marletta, "La trasformazione quadratica (2, 2) fra piani," *Rend. del Circ. Mat. di Palermo,* Vol. XVII (1903), pp. 173–184. "La trasformazioni cubiche (2, 2) fra piani," same volume, pp. 371–385.

§ F. R. Sharpe and V. Snyder, "Types of (2, 2) Point Correspondences Between two Planes." *T. A. M. S.,* Vol. XVIII (1917), pp. 402–414.

‖ R. Baldus, "Zur Theorie der gegenzeitig mehrdeutigen algebraischen Ebenentransformationen," *Mat. Annal.,* Vol. LXXII (1912), pp. 1–56.

2. A general (2, 3) point correspondence is said to exist between two planes $(x)$ and $(x')$ when the points of the two planes are related as follows: Choosing $(x)$ and $(x')$ respectively for the double plane and the triple plane, to a point $P'$ of $(x')$ correspond three points $P_1, P_2, P_3$ of $(x)$, ordinarily distinct. To $P_1, P_2, P_3$ correspond in $(x')$ the original point $P'$ and the residual points $P_1', P_2', P_3'$, respectively, which are usually distinct from each other and from $P'$. One more step will fully illustrate the general procedure. To $P_1', P_2', P_3'$, respectively, correspond the three sets $P_1, P_4, P_5$; $P_2, P_6, P_7$; $P_3, P_8, P_9$, the three points in each set being usually non-coincident.

If in $(x')$ the three residual points $P_1', P_2', P_3'$ coincide in one point $P_0'$, then the three points $P_1, P_2, P_3$ of $(x)$ correspond to either $P'$ or $P_0'$ of $(x')$ and to each of the points $P_1, P_2, P_3$ corresponds the pair $P', P_0'$. Such an involutorial (2, 3) point correspondence may be obtained by combining a (1, 2) and a (1, 3) point correspondence and is known as a (2, 3) compound involution. Since (2, 3) compound involutions, although distinct types, are not general (2, 3) point correspondences, they will not be discussed in this paper.

In the general (2, 3) point correspondence between two planes, a type is distinguished and defined by two equations, each algebraic in the coordinates of both planes. As shown in the next article, such a pair of equations establishes an algebraic relationship between the planes $(x)$ and $(x')$ such that to a point of $(x)$ correspond two points of $(x')$, and to a point of $(x')$ correspond three points of $(x)$. By analogy to Riemann surfaces, $(x)$ is called the double plane and $(x')$ the triple plane. By considering all the possible forms of these defining equations that do not establish (2, 3) compound involutions, twelve independent types of general (2, 3) point correspondences have been obtained. By independent types we mean those that can not be reduced, one to the other, by any series of birational transformations. It will be shown finally that all general (2, 3) point correspondences are birationally equivalent to some one of these twelve types.

3. *General Properties.*—The defining equations of a (2, 3) point correspondence between two planes $(x)$ and $(x')$ may be written as two algebraic equations of the form,

$$\Sigma u_k(x)\, u_k'(x') = 0, \tag{1}$$

$$\Sigma v_k(x)\, v_k'(x') = 0. \tag{2}$$

When $x_1', x_2', x_3'$ are parameters, these equations represent two curves of $(x)$ intersecting in three variable points corresponding to the point $(x_1', x_2', x_3')$,

and when $x_1, x_2, x_3$ are parameters, two curves of $(x')$ intersecting in two variable points corresponding to the point $(x_1, x_2, x_3)$. Connected with each (2, 3) point correspondence, the defining equations establish a transformation between the two planes $(x)$ and $(x')$ such that to a line $C_1(x)$ corresponds a curve $C'_n(x')$ of order $n$ and genus $p'$ which may have fundamental points $A'_j$ of multiplicities $r'_j$. Likewise to a line $C'_1(x')$ corresponds a curve $C_n(x)$ of order $n$ and genus $p$ which may have fundamental points $A_i$ of multiplicities $r_i$.

The fundamental or basis points of a given plane are fixed points on the curves of that plane. The order of a basis point is its multiplicity on the images of general lines. To each basis point of order $r$ in either plane corresponds a basis curve of order $r$ in the other plane. If a curve $C$ passes through a basis point $A$ its image is composite, consisting of the image of $A$ counted as many times as the multiplicity of $A$ on $C$ and a residual curve called the proper image of $C$.

To a point $P'$ of $(x')$ correspond three points $P_1, P_2, P_3$ of $(x)$ which describe the curve $C_n$ as $P'$ describes the line $C'_1$. Similarly, to a point $P$ of $(x)$ correspond two points $P'_1, P'_2$ of $(x')$ which describe $C'_n$ as $P$ describes $C_1$. To the point of intersection of two lines of $(x)$ $[(x')]$ correspond in $(x')$ $[(x)]$ two [three] non-basic intersections of the image curves of these lines. These two [three] image points always lie at the two [three] intersections of the two curves of $(x')$ $[(x)]$ given by the defining equations in which the parameters are the coordinates of the common point of the two lines of $(x)$ $[(x')]$. The images in both planes are distinct except for points on certain fixed curves.

If the two images $P'_1, P'_2$ of a point $P$ of $(x)$ coincide, $P$ is on the branchpoint curve of $(x)$ which will be called $L(x)$. The locus of the corresponding coincidences is the coincidence curve of $(x')$, denoted by $K'(x')$, which is in (1, 1) correspondence with $L(x)$. If in $(x)$, two of the three images of the point $P'$ of $(x')$ coincide, $P'$ is on the branchpoint curve $L'(x')$ and the coincident image points $P_1 \equiv P_2$ describe the coincidence curve $K(x)$ in $(x)$ which is in (1, 1) correspondence with $L'(x')$. At the same time the remaining image point $P_3$ in $(x)$ describes a fixed locus $\Gamma(x)$, the residual image of $L'(x')$ and also in (1, 1) correspondence with it. When all three image points coincide, $K(x)$ and $\Gamma(x)$ have a common tangent at $P_1 \equiv P_2 \equiv P_3$ and $P'$ is a cusp on $L'(x')$. There are but a finite number of points of $(x')$ whose three image points in $(x)$ coincide unless every point of $L'(x')$ has this property.

The equation of $L(x)$ is the condition on the $x_i$ that the two curves of $(x')$ given by the defining equations be tangent, $L(x)$ has no non-basic multiple points. The proper image of $L(x)$ is $K'(x')$ counted twice. $K'(x')$ may have

non-basic multiple points since it is of the same genus as $L(x)$. In like manner the equation of $L'(x')$ is the condition on the $x_i'$ that the two curves of $(x)$ be tangent. Aside from the basis points, the only singularities of $L'(x')$ are cusps. The proper image of $L'(x')$ is $[K(x)]^2\Gamma(x)$. Both $K(x)$ and $\Gamma(x)$ are of the same genus as $L'(x')$ and may have non-basic multiple points. When the genus of the image curves of the straight lines of one plane is known, the order of the branchpoint curve of that plane can be readily found without the above-mentioned algebraic process which is always possible, but sometimes laborious. If $t$ is the multiplicity of the transformation and $p$ the genus, the number of coincidences of the transformation is given by the formula $2(t+p-1)$, known as "Zeuthen's formula."* Since these coincidences are in (1, 1) correspondence with the intersections of the branchpoint curve and a general line of the other plane, the order of that branchpoint curve is $2(t+p-1)$.

All image curves of $(x)$ have only contacts with $L(x)$ in accordance with the lemma:

*If a point $P$ of $(x)$ describe a curve $C$, the necessary and sufficient condition that its images $P_1'$, $P_2'$ of $(x')$ describe distinct curves is that $C$ touch $L(x)$ at every common point.*

This lemma was proved for (2, 2) point correspondences† where it was valid for both planes. In the case of (2, 3) point correspondences, however, it holds only for the double plane $(x)$. For the triple plane $(x')$ we have the lemma:

*If a point $P'$ of $(x')$ describe a curve $C'$, the necessary and sufficient condition that its images $P_1$, $P_2$, $P_3$ of $(x)$ describe distinct curves is that $C'$ and $L'(x')$ have contacts and intersections respectively equal to the intersections of the image of $C'$ with $K(x)$ and $\Gamma(x)$.*

Applying these lemmas to the fixed curves of $(x)$ and $(x')$ we have the theorem:

*In $(x')$ $L'(x')$ and $K'(x')$ have $r$ intersections and $s$ tangencies corresponding in $(x)$ to $r$ tangencies of $L(x)$ and $\Gamma(x)$ and $s$ tangencies of $L(x)$ and $K(x)$.*

If a line $C_1'$ meets $K'(x')$ in $i'$ points, its image in $(x)$ is a curve $C_n$ tangent to $L(x)$ at $i'$ points. The image of $C_n$ is $C_1'$ counted three times, and a

* Zeuthen, "Nouvelle démonstration de théorèmes sur des séries de points correspondants sur deux courbes," *Mat. Annal.*, Vol. III (1871), p. 150.

† F. R. Sharpe and V. Snyder, *loc. cit.*, Article 2.

residual curve that cuts $C_1'$ in $2d+i'$ points corresponding to the $d$ variable double points of $C_n$ and the $i'$ contacts of $L(x)$ and $C_n$. If a line $C_1$ meets $K(x)$ and $\Gamma(x)$ in $i$ and $j$ points respectively, its image in $(x')$ is a curve $C_m'$ which has $i$ contacts and $j$ intersections with $L'(x')$. The image of $C_n'$ is $C_1$ counted twice, and a residual curve intersecting $C_1$ in $2d'+i$ points corresponding to the $i$ contacts of $C_n'$ and $L'(x')$ and the $d'$ variable double points of $C_n'$.

4. *Types of Correspondences.*—There are twelve independent types of general (2, 3) point correspondences between two planes. Special cases may be common to two or more types. Each type is established by a set of defining equations as described in the preceding article. For convenience, these have been divided into three classes with respect to the curve system in the triple plane.

Class 1. Lines and conics; five types.

Class 2. Curves of order $n$ having basis points of multiplicity $n-2$ at the vertex of the line pencil; four types.

Class 3. Conics with two basis points; three types.

The following table shows the curves employed in the defining equations for each type. The symbol $C_n; jP_i$ means a curve of order $n$ with $j$ basis points of each multiplicity $i$.

| Class. | Type. | $u_k'(x')=0.$ | $u_k(x)=0.$ | $v_k'(x')=0.$ | $v_k(x)=0.$ |
|---|---|---|---|---|---|
| 1 | I | line | line | conic | cubic |
| | II | line | cubic | conic | line |
| | III | line | conic; $P_1$ | conic | conic; $P_1$ |
| | IV | line | cubic; $6P_1$ | conic | cubic; $6P_1$ |
| | V | line | conic; $2P_1$ | conic | cubic; $P_2P_1$ |
| 2 | VI | line pencil | cubic | $C_n; P_{n-2}$ | line |
| | VII | line pencil | line pencil | $C_n; P_{n-2}$ | $C_m; P_{m-3}$ |
| | VIII | line pencil | cubic; $6P_1$ | $C_n; P_{n-2}$ | cubic; $6P_1$ |
| | IX | line pencil | cubic; $8P_1$ | $C_n; P_{n-2}$ | $C_9; 8P_3$ |
| 3 | X | conic; $2P_1$ | line | conic; $2P_1$ | cubic |
| | XI | conic; $2P_1$ | conic; $P_1$ | conic; $2P_1$ | conic; $P_1$ |
| | XII | conic; $2P_1$ | cubic; $6P_1$ | conic; $2P_1$ | cubic; $6P_1$ |

Each of these types will now be discussed, Type I in detail and the others more briefly. Similar methods are used in investigating all the types, but the algebraic details naturally differ.

## § 2. CLASS 1. FIVE TYPES.

5. *Type I. Image of a Line.*—The defining equations,

$$\sum_{k=1}^{3} x_k x'_k = 0, \tag{1}$$

$$\sum_{k=1}^{6} u_k(x)\, v'_k(x') = 0, \tag{2}$$

where $u_k(x)=0$, $v'_k(x')=0$ are respectively cubics of $(x)$ and conics of $(x')$, relate the point $(x'_1, x'_2, x'_3)$ to the three intersections of the line and cubic determined by it in $(x)$, and the point $(x_1, x_2, x_3)$ to the two intersections of the line and conic determined by it in $(x')$.

The image of a line, $C'_1 \equiv \sum_{k=1}^{3} a'_k x'_k = 0$, is the quintic

$$C_5 \equiv \sum_{k=1}^{6} u_k(x)\, v'_k(a'_2x_3 - a'_3x_2,\ a'_3x_1 - a'_1x_3,\ a'_1x_2 - a'_2x_1) = 0.$$

$C_5$ has a double point at $(a'_1, a'_2, a'_3)$ and is of genus 5. The image of a line, $C_1 \equiv \sum_{k=1}^{3} a_k x_k = 0$, is the quintic

$$C'_5 \equiv \sum_{k=1}^{6} u_k(a_2x'_3 - a_3x'_2,\ a_3x'_1 - a_1x'_3,\ a_1x'_2 - a_2x'_1)\, v'_k(x') = 0.$$

$C'_5$ has a triple point at $(a_1, a_2, a_3)$ and is of genus 3.

6. *Branchpoint and Coincidence Curves.*—The equation of $L(x)$ is the condition that the line and conic of $(x')$ be tangent. Writing the equation of a tangent to (2) at $(x'_1, x'_2, x'_3)$ and equating the coefficients of this equation to those of (1), we have

$$\sum_{k=1}^{6} u_k \frac{\partial v'_k}{\partial x'_i} = \rho x_i, \qquad i=1, 2, 3. \tag{3}$$

Let $v'_k \equiv a'_k x'^2_1 + b'_k x'^2_2 + c'_k x'^2_3 + 2d'_k x'_2 x'_3 + 2e'_k x'_3 x'_1 + 2f'_k x'_1 x'_2 = 0.$

Eliminating the $x'_i$ and $\rho$ from the three equations of (3) and from (1), we have the equation

$$L(x) \equiv \begin{vmatrix} \Sigma u_k a'_k & \Sigma u_k f'_k & \Sigma u_k e'_k & x_1 \\ \Sigma u_k f'_k & \Sigma u_k b'_k & \Sigma u_k d'_k & x_2 \\ \Sigma u_k e'_k & \Sigma u_k d'_k & \Sigma u_k c'_k & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = 0.$$

$L(x)$ is of order 8 and genus 21 having no singularities. The image of $L_8$ is $K'_{20}$. It is of genus 21 and has 150 double points.

The equation of $L'(x')$ is the condition on the parameters $x'_i$ that the line and cubic of $(x)$ be tangent. It is the equation of the cubic in line coordinates. The coefficients in the equation of the line appear to degree equal to the class

of the cubic and the coefficients of the given curve appear to degree equal to that of the discriminant of a binary cubic.* The cubic is of class 6, the discriminant of order 4, and each element in it is of degree 2 in the $x_i'$. Hence $L'(x')$ is of order 14. $L_{14}'$ has thirty-nine cusps and is of genus 39. The image of $L_{14}'$ is $K_{17}$ counted twice and the residual curve $\Gamma_{36}$. $K_{17}$ and $\Gamma_{36}$ are each of genus 39 and have 81 and 556 double points, respectively.

The order of $L(x)$ by Zeuthen's formula is $2(2+3-1)=8$. The order of $L'(x')$ is $2(3+5-1)=14$. This formula serves as a check when the equations of the branchpoint curves and the genera of the image curves are found by other methods.

$K_{17}$ and $\Gamma_{36}$ have thirty-nine contacts corresponding to the thirty-nine cusps of $L_{14}'$. The three image points corresponding to a point at a cusp of $L_{14}'$ coincide at the corresponding contact of $K_{17}$ and $\Gamma_{36}$. $L_{14}'$ and $K_{20}'$ have sixty-eight contacts corresponding to the contacts of $L_8$ and $K_{17}$, and 144 intersections corresponding to the contacts of $L_8$ and $\Gamma_{36}$.

7. *Successive Images of Lines.*—To a line $C_1'(x')$ corresponds $C_5(x)$ having one variable double point. The image of $C_5$ is $C_{25}'$ which consists of $C_1'$ counted three times, and a residual $C_{22}'$ which passes through the twenty intersections of $C_1'$ and $K_{20}'$ corresponding to the twenty contacts of $C_5$ and $L_8$. The other two intersections of $C_{22}'$ and $C_1'$ correspond to the variable double point of $C_5'$. $C_{22}'$ and $L_{14}'$ intersect in 308 points which consist of eighty-five contacts corresponding to the eighty-five intersections of $C_5$ and $K_{17}$, and 138 intersections corresponding to 138 of the 180 intersections of $C_5$ and $\Gamma_{36}$. To the remaining forty-two intersections of $C_5$ and $\Gamma_{36}$ correspond the forty-two intersections of $L_{14}'$ and $C_1'$ counted three times.

To a line $C_1(x)$ corresponds $C_5'(x')$ having one variable triple point. The image of $C_5'$ is $C_{25}$ consisting of $C_1$ counted twice and a residual $C_{23}$. Of the twenty-three intersections of $C_1$ and $C_{23}$, six correspond to the triple point on $C_5'$ and the remaining seventeen are at the seventeen intersections of $K_{17}$ and $C_1$ which correspond to the seventeen contacts of $C_5'$ and $L_{14}'$. To the remaining thirty-six intersections of $C_5'$ and $L_{14}'$ correspond the thirty-six intersections of $\Gamma_{36}$ and $C_1$, which are not on $C_{23}$. $C_{23}$ and $L_8$ intersect in 184 points which are ninety-two contacts corresponding to ninety-two of the 100 intersections of $C_5'$ and $K_{20}'$. To the remaining eight intersections of $C_5'$ and $K_{20}'$ correspond the sixteen intersections of $C_1$ counted twice and $L_8$.

Two lines $C_1'$, $\overline{C}_1'$ of $(x')$ have $C_5$, $\overline{C}_5$ for images in $(x)$ which intersect in twenty-five points. Three of these are collinear and correspond to the common

* See Salmon's "Higher Plane Curves," 4th ed., Art. 91, 188, 222.

point of $C_1'$, $\overline{C}_1'$. The remaining twenty-two points have for their images the twenty-two intersections of $C_1'$ with $\overline{C}_{22}'$ and $\overline{C}_1'$ with $C_{22}'$. The images of two lines $C_1$, $\overline{C}_1$ of $(x)$ are $C_5'$, $\overline{C}_5'$ of $(x')$ which intersect in twenty-five points. Two of these intersections correspond to the common point of $C_1$, $\overline{C}_1$, and the remaining twenty-three have for their images the twenty-three intersections of $C_1$ with $\overline{C}_{23}$, and $\overline{C}_1$ with $C_{23}$, and also twenty-three intersections of $C_{23}$ with $\overline{C}_{23}$.

8. *Cubics in* $(x)$ *with One Basis Point.*—Let $P_1$ be a simple basis point on the cubics of $(x)$. The image of $C_1'$ is $C_5$ passing through $P_1$ and having a variable double point. The image of $C_1$ is $C_5'$ which has a variable triple point. The image of $P_1$ is the fixed line $f_1'$. The image of $f_1'$ is $P_1$ and a residual $f_5$ with a triple point at $P_1$. $K_{17}$ and $f_5$ have the same three tangents at $P_1$.

$L(x)$ is of order 8 and genus 20, having a double point at $P_1$. Its proper image is $K_{19}'$. $L'(x')$ is of order 14 and genus 39 having thirty-nine cusps. $K_{17}$ has a triple point and $\Gamma_{36}$ an eight-fold point at $P_1$.

A line $C_1$ through $P_1$ corresponds to $f_1'$ and $C_4'$. $C_4'$ has a variable double point on $f_1'$ and is of genus 2. The two simple intersections of $f_1'$ and $C_4'$ are images of the direction of $C_1$ through $P_1$. The image of $C_4'$ is $C_1$ counted twice and a residual $C_{18}$ with a double point at $P_1$. $C_{18}$ meets $C_1$ in sixteen points apart from $P_1$, fourteen of which are at the non-basic intersections of $C_1$ and $K_{17}$, and two correspond to the variable double point of $C_4'$.

9. *Conics in* $(x)$ *with One Basis Point.*—Let the simple basis point be $P_1'$. The image of $C_1'$ is $C_5$ which has a variable double point. The image of $C_1$ is $C_5'$ passing through $P_1'$ and with a variable triple point. The image of $P_1'$ is the fundamental line $f_1$. To $f_1$ corresponds $P_1'$ and a residual $f_5'$ with a four-fold point at $P_1'$. $K_{20}'$ and $f_5'$ have the same four tangents at $P_1'$. $L(x)$ is of order 8 and genus 21. $K_{20}'$ has a four-fold point at $P_1'$ and 144 double points. $L'(x')$ is of order 14 with a four-fold point at $P_1'$. It is of genus 36 and has thirty-six cusps. $K_{16}$ and $\Gamma_{34}$ have 66 and 487 non-basic double points, respectively.

To a line $C_1'$ through $P_1'$ corresponds $f_1$ and a $C_4$ of genus 3. Two of the intersections of $f_1$ and $C_4$ correspond to the direction of $C_1'$ through $P_1'$. The image of $C_4$ is $C_1'$ counted three times and a residual $C_{17}'$ passing through $P_1'$ simply. $C_{17}'$ meets $C_1'$ in sixteen points apart from $P_1'$ which are at the sixteen non-basic intersections of $K_{20}'$ and $C_1'$.

10. *Cubics in* $(x)$ *with* $i \leqq 9$ *Basis Points and Conics in* $(x')$ *with* $j \leqq 4$ *Basis Points.*—In general there may be any combination of $i$ simple basis points, $i \leqq 9$, on the cubics of $(x)$, and $j \leqq 4$ on the conics of $(x')$, and any par-

ticular case may be obtained from the following results by giving to $i$ and $j$ their appropriate values. Let the basis points of $(x)$ be denoted by $P_i$ and those of $(x')$ by $P'_j$. The image of $C'_1$ is $C_5$ passing simply through each of the $P_i$ of genus 5 and having a variable double point. The image of $C_1$ is $C'_5$ passing simply through each of the $P'_j$ of genus 3 and having a variable triple point. To each basis point $P_i$ corresponds a fixed line $f'_i$. To each $f'_i$ corresponds the $P_i$ whose image it is, and a residual $f_5$ with a triple point at that $P_i$ and passing simply through the remaining $i-1$ $P_i$. To each $P'_j$ corresponds a fixed line $f_j$. The image of $f_j$ is the $P'_j$ to which it corresponds and a residual $f'_5$ having a four-fold point at that $P'_j$ and passing simply through the the remaining $j-1$ $P'_j$.

$L(x)$ is of order 8 and genus $21-i$, having a double point at each of the $P_i$. $K'_{20-i}$ has each of the $P'_j$ of multiplicity 4 and $\frac{1}{2}(i-20)(i-15)-12j$ non-basic double points. $L'_{14}$ is of genus $39-3j$ having $39-3j$ cusps and a four-fold point at each of the $P'_j$. $K_{17-j}$ has a triple point at each of the $P_i$ and $\frac{1}{2}(j^2-25j-6i+162)$ double points. $\Gamma_{36-2j}$ has an eight-fold point at each of the $P_i$ and $2(j^2-33j-14i+278)$ double points. Apart from the $P'_j$, $L'_{14}$ and $K'_{20-i}$ have $68-3i-4j$ contacts and $144-8i-8j$ intersections corresponding to the non-basic contacts of $L_8$ with $K_{17-j}$ and $\Gamma_{36-2j}$, respectively.

The images of a general line through one basis point of either plane are similar to those in the cases previously discussed. The lines joining two or more basis points will now be considered. Since the cubics and conics are not composite, there can be no more than three collinear basis points in $(x)$ and two in $(x')$. The image of the line joining two basis points $P_1$, $P_2$ of $(x)$ consists of $f'_1$, $f'_2$ and a fixed $C'_3$ through their intersection and through all the $P'_j$. Let three basis points $P_1$, $P_2$, $P_3$ be collinear. To the line joining them corresponds $f'_1$, $f'_2$, $f'_3$ all concurrent and a fixed $C'_2$ not through their common intersection, but through the $P'_j$. The image of the line joining two basis points $P'_1$, $P'_2$ of $(x')$ consists of $f_1$, $f_2$ and a fixed $C_3$ not through their intersection, but through all the $P_i$.

11. *Line Pencil in Each Plane.*—Let the lines in $(x)$ given by equation (1) form a pencil whose vertex $P\equiv(0,0,1)$ is not on the cubics. Then the lines of $(x')$ also form a pencil whose vertex $P'\equiv(0,0,1)$ can not be on the conics. The image of $C'_1$ is $C_5$ with a double point at $P$ and of genus 5. To $C_1$ corresponds $C'_5$ with a triple point at $P'$ and of genus 3. The image of $P'$ is the cubic $f_3$ whose image in $(x')$ is $P'$ and a residual $f'_{15}$ with a twelve-fold point at $P'$. The image of $P$ is the conic $f'_2$ to which corresponds $P$ and a residual $f_{10}$ with a six-fold point at $P$. To the line $p'$ through $P'$ corresponds

$f_3$ and the line $p$ counted twice through $P$. To $p$ corresponds $f_2'$ and $p'$ counted three times. The rays of the two pencils are in (1, 1) correspondence. $L(x)$ is of order 8 and genus 20 having a double point at $P$. $K_{18}'$ has a twelve-fold point at $P'$ and fifty double points. $L_{14}'$ has thirty cusps, a six-fold point at $P'$ and is of genus 33. $K_{14}$ has a six-fold point at $P$ and thirty double points. $\Gamma_{24}$ has a sixteen-fold point at $P$ and 100 double points.

12. Now assume that both defining equations are satisfied by $x_2=0$, $x_1'=0$. The image of $C_1'$ is $x_2=0$ and $C_4$ of genus 3 through $P$. The image of $C_1$ is $x_1'=0$ and $C_4'$ with a double point at $P'$ and of genus 2. To $P'$ corresponds $x_2=0$ and a conic $f_2$ not through $P$. To $P$ corresponds $x_1'=0$ and the line $f_1'$ not through $P'$. The image of $x_2=0$ is $x_1'^3=0$ and $f_1'$. The image of $x_1'=0$ is $x_2^2=0$ and $f_2$. $L_8$ has $x_2^2=0$ as a component, the proper curve being a sextic of genus 10 and not through $P$. $K_{18}'$ consists of $x_1'^6=0$ and a proper $C_{12}'$ with a six-fold point at $P'$. $L_{14}'$ has $x_1'^4=0$ as a component and a proper $C_{10}'$ with a double point at $P'$, of genus 17 and having eighteen cusps. $K_{14}$ consists of $x_2^4=0$ and a $C_{10}$ with a double point at $P$. $\Gamma_{24}$ consists of $x_2^8=0$ and a $C_{16}$ with a six-fold point at $P$. The fixed components $x_2=0$, $x_1'=0$ are illustrations of the curves $D_1$, $D_2$ discussed by R. Baldus.*

13. When, in addition to the line pencils, there are $i \leqq 7$ basis points on the cubics of $(x)$ and $j \leqq 3$ basis points of the conics of $(x')$, the image of $C_1'$ is $C_5$ with a double point at $P$ and simple points at each of the $P_i$, and the image of $C_1$ is $C_5'$ with a triple point at $P'$, and passing simply through each of the $P_j'$. $L_8$ has double points at $P$ and at each of the $P_i$. $K_{18-i}'$ has a $(12-i)$-fold point at $P$ and triple points at each of the $P_j'$. $L_{14}'$ has a six-fold point at $P'$ and four-fold points at each of the $P_j'$. $K_{14-j}$ has a $(6-j)$-fold point at $P$ and double points at each of the $P_i$. $\Gamma_{24-2j}$ has a $(16-2j)$-fold point at $P$ and four-fold points at each of the $P_i$. The image of $P'$ is $f_3$ through the $P_i$, but not through $P$. The image of $P$ is $f_2'$ through the $P_j'$, but not through $P'$. To each $P_j'$ corresponds a line through $P$, and to each $P_i$ a line through $P'$.

14. *Geometric Depiction of Pencil Cases.*—All the cases in which the bilinear equation has but two terms can be visualized after the method of Marletta† by means of a quintic surface in ordinary space. Let $F=0$ be a surface of order 5 with a triple point at $P_3$ and a double point at $P_2$. Choose two planes $(x)$ and $(x')$ not through $P_2$ or $P_3$ as the double and triple plane, respectively, upon which the correspondence is to be pictured. Take any point $P$ in $(x)$ and any point $P'$ in $(x')$. Connect $P$ with $P_3$. The line $PP_3$ meets $F$ in

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* Baldus, *loc. cit.*, 2, Articles 6, 7, 8.

† G. Marletta, *loc. cit.*

two points not at $P_3$. Project these two points from $P_2$ upon $(x')$ obtaining in $(x')$ two points that are the images of the point $P$ in $(x)$. Going the other way, join $P'$ with $P_2$ and project the three points of intersection of $P'P_2$ and $F$, not at $P_2$, on $(x)$ from $P_3$. This gives in $(x)$ the three images of $P'$.

The apparent contour of $F$ from $P_2$, *i. e.*, the tangent cone, is cut by $(x')$ in the curve $L'_{14}$. Every tangent line, elements of the tangent cone from $P_2$, meets $F$ in one other point. The projections of the locus of the points of tangency and the locus of the residual intersections upon $(x)$ from $P_3$ give the curves $K_{14}$ and $\Gamma_{24}$, respectively. In like manner the tangent cone to $F$ from $P_3$ is cut by $(x)$ in $L_8$, and its curve of contact is projected upon $(x')$ from $P_2$ into $K'_{18}$.

If $x_2=0$, $x'_1=0$ satisfy both defining equations, the surface $F_5$ consists of a plane through $P_2P_3$ and a residual surface $F_4$ passing through $P_2$ and having a double point at $P_3$. The depiction on $(x)$ and $(x')$ may now be obtained as before. Basis points in $(x)$ and $(x')$ are accounted for by fixed lines on $F$ passing through $P_2$ and $P_3$ respectively.

15. *Type II.*—The defining equations of this type are obtained by interchanging the parameters of the line and conic of $(x')$. The image of a line, $C'_1 \equiv \sum_{k=1}^{3} a'_k x'_k = 0$, is,

$$C_7 \equiv \sum_{k=1}^{3} x_k v'_k (a'_2u_3 - a'_3u_2,\ a'_3u_1 - a'_1u_3,\ a'_1u_2 - a'_2u_1) = 0.$$

Since the three cubics

$$u_1/a'_1 = u_2/a'_2 = u_3/a'_3$$

form a pencil, their nine intersections are variable double points on $C_7$, leaving it of genus 6. The image of $C_1$ is $C'_7$ with four variable triple points at the common points of the pencil of conics whose parameters are those of $C_1$. $C'_7$ is of genus 3.

The equation of $L(x)$ is found as in Type I. $L_8$ is of genus 21 and has no singularities. $K'_{28}$ is of genus 21 and has 330 double points. As in Type I we find that $L'_{16}$ has fifty-seven cusps and is of genus 48. $K_{28}$ and $\Gamma_{66}$ have 183 and 2032 double points respectively.

16. *Type III.*—The defining equations are

$$\sum_{k=1}^{5} u_k(x) v'_k(x') = 0, \tag{1}$$

$$\sum_{k=1}^{3} v_k(x) x'_k = 0, \tag{2}$$

wherein $u_k(x)=0$, $v_k(x)=0$ are conics of $(x)$ having a basis point at $R\equiv(1,0,0)$ and $v'_k(x')=0$ are general conics of $(x')$. To $C'_1$ corresponds $C_6$ with a triple point at $R$ and three variable double points at the other three intersections of a pencil of conics whose parameters are those of $C'_1$. $C_6$ is of genus 4. The image of $C_1$ is $C'_6$ with seven variable double points and of genus 3. If $C_1$ passes through $R$ its proper image is a cubic of genus 1. The image of of $R$ is a fixed cubic of genus 1.

$L_8$ has a four-fold point at $R$ and is of genus 15. $K'_{18}$ has 121 double points. The equation of $L'(x')$ is the condition that two conics through a fixed point have a contact not at that point. This condition is of order 4 in the coefficients of each of the conics. Then $L'(x')$ is of order 12. It has twenty-seven cusps and is of genus 28. $K_{20}$ has $R$ of multiplicity 11, and eighty-eight double points. $\Gamma_{32}$ has a fourteen-fold point at $R$ and 346 double points.

17. *Type IV.*—The correspondence is defined by equations of the same form as those in Type III, wherein now $u_k(x)=0$, $v_k(x)=0$ are cubics through six simple basis points $P_i$ of $(x)$. The image of $C'_1$ is $C_9$ with each of the $P_i$ three-fold, and with double points at the three variable intersections of a pencil of cubics whose parameters vary with $C'_1$. To $C_1$ corresponds $C'_9$ with twenty-three variable double points. If $C_1$ passes through one $P_i$ its proper image is $C'_6$. To each $P_i$ corresponds a fundamental cubic. If $C_1$ contains two $P_i$ its proper image is a fixed $C'_3$. The proper image of a cubic through the six $P_i$ is a $C'_9$.

$L_{12}$ has a four-fold point at each $P_i$ and is of genus 19. $K'_{18}$ has 117 double points. The equation of $L'(x')$ is the condition that the two cubics through the $P_i$ be tangent. This condition is of order 6 in the coefficients of each. $L'_{18}$ has seventy-five cusps and is of genus 61. $K_{33}$ has an eleven-fold point at each $P_i$ and 105 double points. $\Gamma_{96}$ has a thirty-two-fold point at each $P_i$ and 1428 double points.

18. *Type V.*—The defining equations are written as in Type III. For the present type $u_k(x)=0$ represent cubics with a double point at $P\equiv(0,0,1)$ and a simple point at $R\equiv(1,0,0)$ and $v_k(x)=0$ are conics through $P$ and $R$. The image of $C'_1$ is $C_7$ with $P$ four-fold, $R$ three-fold and two double points at the two variable intersections of a pencil of conics through $P$ and $R$ whose parameters vary with $C'_1$. $C_7$ is of genus 4.

In order to study the properties of the image of $C_1$, we may write the defining equations as follows:

$$\sum_{k=1}^{6} u_k(x)v'_k(x') \equiv x_2(a'x_1^2+b'x_2x_3+c'x_1x_3+e'x_1x_2+f'x_2^2)+d'x_1^2x_3=0, \quad (1)$$

$$\sum_{k=1}^{3} v_k(x)x'_k \equiv x_2(p'x_1+q'x_2+r'x_3)+s'x_1x_3=0, \quad (2)$$

wherein $a'$, $b'$, $c'$, $d'$, $e'$, $f'$ are quadratic and $p'$, $q'$, $r'$, $s'$ linear in the $x'_i$. Multiply (1) by $s'$, (2) by $d'x_1$ and subtract, using the resulting equation

$$s'(a'x_1^2+b'x_2x_3+c'x_1x_3+e'x_1x_2+f'x_2^2)-d'x_1(p'x_1+q'x_2+r'x_3)=0, \quad (3)$$

with (2) as the pair of defining equations. Eliminate the $x_i$ from the equation of $C_1$ and (2) and (3) and we obtain a $C'_8$ with double points at the two intersections of $s'=0$, $d'=0$ and containing $s'=0$ as a fixed component. Then the proper image of $C_1$ is $C'_7$, which passes simply through the two basis points $s'=0$, $d'=0$. It is of genus 4 and has eleven variable double points.

The proper image of a line through $P$ is a cubic which does not pass through $s'=0$, $d'=0$. The image of $P$ is $f'_4$. The proper image of a line through $R$ is a quartic not through $s'=0$, $d'=0$. To $R$ corresponds $f'_3$. Both $f'_4$ and $f'_3$ pass through $s'=0$, $d'=0$. The image of the fundamental line $PR$ is $f'_3f'_4$. The image curves of $(x)$ intersect $PR$ only at the basis points $P$ and $R$. Every point of $PR$ except $P$ and $R$ corresponds to the two basis points $s'=0$, $d'=0$. To each of the basis points $s'=0$, $d'=0$ corresponds the line $PR$. The image of $s'=0$ is $PR$ counted twice, and a fixed $C_5$ with a triple point at $P$ and a double point at $R$. The image of $d'=0$ is $PR$ counted twice and a fixed $C_{12}$, with $P$ seven-fold and $R$ five-fold.

$L_{10}$ has $P$ six-fold and $R$ four-fold and is of genus 15. $K'_{17}$ has 105 double points. The condition that the cubic and conic of $(x)$ defined by (1) and (2) be tangent is of order 4 in the coefficients of each, giving $L'(x')$ of order 12. $L'_{12}$ has twenty-seven cusps and is of genus 28. Neither $L'_{12}$ nor $K'_{17}$ pass through the basis points $s'=0$, $d'=0$. $K_{28}$ has a fourteen-fold point at $P$, a nine-fold point at $R$ and seventy-six double points. $\Gamma_{38}$ has a twenty-fold point at $P$, an eighteen-fold point at $R$ and 295 double points.

## § 3. Class 2. Four Types.

19. *Type VI.*—The defining equations are

$$\sum_{k=1}^{3} x_k\psi'_k(x')=0, \quad (1)$$

$$x'_1u_1(x)+x'_3u_3(x)=0, \quad (2)$$

wherein $u_1(x)=0$, $u_3(x)=0$ are cubics of $(x)$, and

$$\psi'_k(x') \equiv x'^2_2u'_k(x'_1, x'_3)+x'_2v'_k(x'_1, x'_3)+w'_k(x'_1, x'_3)=0,$$

($u'_k$, $v'_k$, $w'_k$ being homogeneous functions of $x'_1$, $x'_3$ of the respective degrees $n-2$, $n-1$, $n$) are curves of order $n$ with $(n-2)$-fold points at $Q' \equiv (0,1,0)$, the vertex of the line pencil of $(x')$. The cubics of $(x)$ would have no basis points but for the fact that the lines of $(x')$ form a pencil. This forces the cubics of $(x)$ to form a pencil, thus introducing nine simple basis points $P_i$ into $(x)$.

The image of $C'_1$ is $C_{3n+1}$ with an $n$-fold point at each $P_i$ and of genus $3n$. If $C'_1$ passes through $Q'$ its proper image is a cubic counted twice through the $P_i$. The image of $Q'$ is $f_{3n-5}$ with an $(n-2)$-fold point at each $P_i$. To $C_1$ corresponds $C'_{3n+1}$ with a $(3n-5)$-fold point at $Q'$, and a triple point at each of the $4n-4$ variable intersections of a pencil of curves of order $n$ whose parameters vary with $C_1$. $C'_{3n+1}$ is of genus $3n-3$. If $C_1$ passes through a $P_i$, its proper image is $C'_{2n+1}$ with $Q'$ of multiplicity $2n-3$. If $C_1$ passes through two $P_i$ its proper image is $C'_{n+1}$ with an $(n-1)$-fold point at $Q'$. The image of each $P_i$ is $f'_i$ of order $n$ with $Q'$ $(n-2)$-fold. The proper image of a cubic through the $P_i$ is a line counted three times through $Q'$. The line pencil of $(x')$ is in $(1,1)$ correspondence with the pencil of cubics of $(x)$.

To find the equation of $L(x)$, solve equation (2) for $x'_3$ in terms of $x'_1$, substitute this value of $x'_3$ in (1) and rearrange, obtaining a quadratic in $x'_2/x'_3$ whose discriminant equated to zero is $L(x)$. $L(x)$ is of order $6n-4$ and genus $12n-12$ with a $(2n-2)$-fold point at each $P_i$. $K'(x')$ is of order $6n-2$ with a $(6n-8)$-fold point at $Q'$ and $18n-18$ double points. $L'(x')$, found as in Type I, is of order $6n+4$, genus $30n-15$, has $Q'$ $(6n-8)$-fold and $36n-18$ cusps. $K(x)$ is of order $12n-12$ and has a $(4n-2)$-fold point at each $P_i$ and $18n-6$ double points. $\Gamma(x)$ is of order $48n-32$ and has a $(16n-12)$-fold point at each $P_i$ and $162n-126$ double points. Apart from $Q'$, $L'_{6n-4}$ and $K'_{6n-2}$ have $24n-14$ contacts and $60n-44$ intersections corresponding to the non-basic contacts of $L_{6n-4}$ with $K_{12n-2}$ and $\Gamma_{48n-32}$ respectively.

20. *Type VII.*—The defining equations are

$$\sum_{k=1}^{t} \phi_k(x)\psi'_k(x') = 0, \tag{1}$$

$$x_1x'_1 + x_2x'_3 = 0. \tag{2}$$

$$\phi_k(x) \equiv x_3^2 u_k(x_1, x_2) + x_3^2 v_k(x_1, x_2) + x_3 w_k(x_1, x_2) + s_k(x_1, x_2) = 0$$

(wherein $u_k$, $v_k$, $w_k$, $s_k$ are homogeneous functions of the respective degrees $m-3$, $m-2$, $m-1$, $m$ in $x_1$, $x_2$) is a $C_m$ with an $(m-3)$-fold point at $P \equiv (0,0,1)$, the vertex of the line pencil of $(x)$. $\psi'_k(x')$ is defined as in Type VI. Equation (1) has $t$ terms where $t$ takes the value $3n$ or $4m-2$

according as $m$ is greater or less than $1/4(3n+2)$. A special case of this type for $n=2$, $m=3$ has been discussed in Article 11.

The image of $C_1'$ is $C_{m+n}$ with an $(m+n-3)$-fold point at $P$ and of genus $2(m+n)-5$. The image of $C_1$ is $C'_{m+n}$ with an $(m+n-2)$-fold point at $Q'$ and of genus $m+n-2$. The proper image of a line $p$ through $P$ is a line $q'$ counted three times through $Q'$, and the proper image of $q'$ is $p$ counted twice. The pencils of $(x)$ and $(x')$ are in $(1,1)$ correspondence. To $P$ corresponds $f'_{m+n-3}$ with $Q'$ $(m+n-5)$-fold. To $Q'$ corresponds $f_{m+n-2}$ with an $(m+n-5)$ fold point at $P$.

$L(x)$, obtained as in Type VI, is of order $2(m+n)-2$ and genus $10(m+n)-30$, having a $[2(m+n)-8]$-fold point at $P$. $K'(x')$ is of order $6(m+n)-12$ having $Q'$ of multiplicity $6(m+n)-18$ and $20(m+n)-50$ double points. To obtain the equation of $L'(x')$, solve equation (2) for $x_2$ in terms of $x_1$ and substitute this value of $x_2$ in (1). Then (1) may be written as a cubic in $x_3/x_1$ whose discriminant equated to zero is the equation of $L'(x')$. It is of order $4(m+n)-6$ and genus $16(m+n)-47$ with $Q'$ of multiplicity $4(m+n)-14$ and $12(m+n)-30$ cusps. $K(x)$ is of order $4(m+n)-6$ with $P$ of multiplicity $4(m+n)-14$ and $12(m+n)-30$ double points. $\Gamma(x)$ is of order $8(m+n)-16$ with $P$ of multiplicity $8(m+n)-24$ and $40(m+n)-100$ double points.

21. *Notation.*—In the types of Classes 2 and 3 the defining equations represent respectively the same curve systems in the double plane as in Class 1. Thus the discussion of the succeeding types is so similar to the corresponding types of the first class that only a tabulation of the results is essential in most cases. The following notation will be used in this tabulation:

The symbol "$\sim$" meaning "corresponds to";

$L$, $K'$, $L'$, $K$, $\Gamma$, fixed curves as defined heretofore;

$f$, $f'$, fundamental curves of $(x)$ and $(x')$;

$C$, $C'$, variable curves of $(x)$ and $(x')$;

$P$, $Q$, $R$, $P'$, $Q'$, $R'$, basis points of $(x)$ and $(x')$;

$\overline{P}$, non-basic but fixed points of either plane;

$\overline{\overline{P}}$, variable points of either plane;

$p$, genus of curve being described;

$k$, cusps of $L'$;

Subscripts of curves denote their order;

Subscripts of points denote their multiplicity on the curve being described; (the subscripts $i$ and $j$, however, denote any one of a given number of basis points or curves playing the same rôle).

The following types will illustrate the use of this notation:

22. *Type VIII.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)\,\psi'_k(x') = 0, \tag{1}$$

$$x'_1 v_1(x) + x'_3 v_3(x) = 0. \tag{2}$$

in which $u_k(x)=0$, $v_1(x)=0$, $v_3(x)=0$, are cubics with six simple basis points $P_i$. The cubics of equation (2) form a pencil because their parameters belong to a line pencil, and thus introduce three additional basis points $Q_j$ into $(x)$. The pencil of cubics of $(x)$ and the line pencil of $(x')$ are in (1, 1) correspondence.

$C'_1 \sim C_{3n+3}$, $p=3n+1$; $6P_{n+1}$, $3Q_n$.
$C_1 \sim C'_{3n+3}$, $p=3n-1$; $Q'_{3n-3}$, $(12n-4)\overline{\overline{P}}_2$.
$P_i \sim f'_{i,\,n+1}$; $Q'_{n-1}$.
$Q_j \sim f'_{j,\,n}$; $Q'_{n-2}$.
$Q' \sim f_{3n-3}$; $6P_{n-1}$, $3Q_{n-2}$.
$L_{6n}$, $p=12n-8$; $6P_{2n}$, $3Q_{2n-2}$.
$K'_{6n}$, $p=12n-8$; $Q'_{6n-6}$, $(18n-12)\overline{P}_2$.
$L'_{6n+6}$, $p=30n-15$; $Q'_{6n-6}$, $(36n-6)k$.
$K_{12n+6}$, $p=30n-15$; $6P_{4n+2}$, $3Q_{4n-2}$, $18n\overline{P}_2$.
$\Gamma_{48n-12}$, $p=30n-15$; $6P_{16n-4}$, $3Q_{16n-8}$, $(167n-72)\overline{P}_2$.

23. *Type IX.*—The defining equations are written as in Type VIII. For the present type, $v_1(x)=0$, $v_3(x)=0$ represent cubics of $(x)$ passing through eight basis points $P_i$ and therefore determining a ninth $Q$; $u_k(x)=0$ represents curves of order 9 with triple points at each of the eight $P_i$, but not passing through $Q$.

$C'_1 \sim C_{3n+9}$, $p=3n+4$; $8P_{n+3}$, $Q_n$.
$C_1 \sim C'_{3n+9}$, $p=3n+5$; $Q'_{3n+3}$, $(12n+20)\overline{\overline{P}}_2$.
$Q' \sim f_{3n+3}$; $8P_{n+1}$, $Q_{n-2}$.
$P_i \sim f'_{i,\,n+3}$; $Q'_{n+1}$.
$Q \sim f'_n$; $Q'_{n-2}$.
$L_{6n+12}$, $p=12n+4$; $8P_{2n+4}$, $Q_{2n-2}$.
$K'_{6n+6}$, $p=12n+4$; $Q'_{6n}$, $(18n+6)\overline{P}_2$.
$L'_{6n+12}$, $p=30n+25$; $Q'_{6n}$, $(36n+30)k$,
$K_{12n+30}$, $p=30n+25$; $8P_{4n+10}$, $Q_{4n-2}$, $(18n+18)\overline{P}_2$.
$\Gamma_{48n+48}$, $p=30n+25$; $8P_{16n+16}$, $Q_{16n+4}$, $(162n+90)\overline{P}_2$.

## § 4. Class 3. Three Types.

24. *Type X.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)c'_k(x')=0, \tag{1}$$

$$\sum_{k=1}^{8} x_k v'_k(x')=0, \tag{2}$$

wherein $c'_k(x')=0$, $v'_k(x')=0$ are conics of $(x')$ with basis points $Q'\equiv(0,1,0)$ and $R'\equiv(1,0,0)$ and $u_k(x)=0$ are general cubics of $(x)$. The image of $C_1$ is $C'_8$ with four-fold points at $Q'$ and $R'$ and triple points at the two variable intersections of a pencil of conics whose parameters depend on those of $C_1$. $C'_8$ is of genus 3. To obtain the image of $C'_1$ we may write the defining equations as follows:

$$ax'^2_3+bx'_2x'_3+cx'_1x'_3+dx'_1x'_2=0, \tag{1}$$

$$px'^2_3+qx'_2x'_3+rx'_1x'_3+sx'_1x'_2=0, \tag{2}$$

wherein $a$, $b$, $c$, $d$ and $p$, $q$, $r$, $s$ are respectively general cubic and linear functions of $x_1$, $x_2$, $x_3$. Multiply (1) by $s$, (2) by $d$, subtract (2) from (1), remove the common factor $x_3$ and we obtain the equation

$$(cs-rd)x'_1+(bs-qd)x'_2+(as-pd)x'_3=0. \tag{3}$$

Using (3) with either (1) or (2) as defining equations, the image of $C'_1$ is found to be $C_9$ with double points at the sixteen intersections of a pencil of quartics whose parameters vary with $C'_1$. Three of these double points lie at the intersections of $s=0$, $d=0$, and the remaining thirteen are variable. The line $s=0$ is a fixed component of $C_9$ so the proper image of $C'_1$ is $C_8$ passing through the three basis points $s=0$, $d=0$ and having thirteen variable double points. $C_8$ is of genus 8.

$L_8$, $p=21$.
$K'_{32}$, $p=21$; $Q'_{16}$, $R'_{16}$, $(204)\overline{P}_2$.
$L'_{20}$, $p=39$; $Q'_{10}$, $R'_{10}$, $42k$.
$K_{18}$, $p=39$; $97\overline{P}_2$.
$\Gamma_{44}$, $p=39$; $864\overline{P}_2$.

25. The following statements hold for all three types of Class 3. The images of $Q'$ and $R'$ are respectively the basis curves

$$bs-dq=0, \quad cs-rd=0.$$

The line $Q'R'$ is the image of each of the three basis points $s=0$, $d=0$ of $(x)$. All the curves of $(x')$ intersect $Q'R'$ only in points $Q'$ and $R'$ which are always points of equal multiplicity. There is no proper image of the line $Q'R'$, but

every point of it except $Q'$ and $R'$ corresponds to the three basis points $s=0$, $d=0$. The equation of $L(x)$ is the condition that the two conics of $(x')$ be tangent, which is of order 2 in the coefficients of each. The curves $L(x)$, $K(x)$, $\Gamma(x)$ do not pass through the basis points $s=0$, $d=0$.

26. *Type XI.*—The defining equations are

$$\sum_{k=1}^{4} u_k(x)c'_k(x')=0, \tag{1}$$

$$\sum_{k=1}^{4} v_k(x)v'_k(x')=0, \tag{2}$$

wherein $u_k(x)=0$, $v_k(x)=0$ are conics with one basis point $P$.

$C'_1 \sim C_8$, $p=6; P_4$, $(s=0, d=0)_1$, $9\overline{\overline{P}}_2$.
$C_1 \sim C'_8$, $p=3; Q'_4, R'_4, 6\overline{\overline{P}}_2$.
$Q' \sim f_4; P_2$, $(s=0, d=0)_1$.
$R' \sim f_4; P_2$, $(s=0, d=0)_1$.
$P \sim f'_4; Q'_2, R'_2$.
$L_8$, $p=15; P_4$.
$K'_{24}$, $p=15; Q'_{12}, R'_{12}, 106\overline{P}_2$.
$L'_{16}$, $p=25; Q'_8, R'_8, 24k$.
$K_{18}$, $p=25; P_{10}, 66\overline{P}_2$.
$\Gamma_{28}$, $p=25; P_{12}, 260\overline{P}_2$.

27. *Type XII.*—In the defining equations, $u_k(x)=0$, $v_k(x)=0$ are cubics with six simple basis points $P_i$.

$C'_1 \sim C_{12}$, $p=10; 6P_4$, $(s=0, d=0)_1$, $9\overline{\overline{P}}_2$.
$C_1 \sim C'_{12}$, $p=5; Q'_6, R'_6, 20\overline{\overline{P}}_2$.
$P_i \sim f'_{i,4}; Q'_2, R'_2$.
$Q' \sim f_6; 6P_2$, $(s=0, d=0)_1$.
$R' \sim f_6; 6P_2$, $(s=0, d=0)_1$.
$L_{12}$, $p=19; 6P_4$.
$K'_{24}$, $p=19; Q'_{12}, R'_{12}, 102\overline{P}_2$.
$L'_{24}$, $p=55; Q'_{12}, R'_{12}, 66k$.
$K_{30}$, $p=55; 6P_{10}, 81\overline{P}_2$.
$\Gamma_{84}$, $p=55; 6P_{28}, 1080\overline{P}_2$.

## § 5. Completeness of the Classification.

28. *Reduction of Certain Cases to Type Form.*—It remains now to be proved that all cases of general (2, 3) point correspondences are birationally equivalent to certain of these twelve independent types. All cases where the

defining equations are made up of combinations of the curve systems used in the twelve types (except those that define (2, 3) compound involutions) either belong to these types or can be reduced to certain of them by quadratic transformations. These curve systems are as follows: In $(x')$ lines and conics; line pencil and curves of order $n$ with the vertex of the pencil $(n-2)$-fold; conics with two basis points. In $(x)$ lines and cubics; line pencil and curves of order $m$ with the vertex of the pencil $(m-3)$-fold; conics with one basis point; cubics with six basis points; cubics with a double and a simple basis point through which pass conics; cubics with eight basis points and curves of order 9 with triple points at each of them.

In Class 1, all the cases in which the lines of $(x')$ form a pencil are particular cases of the corresponding types of Class 2 for $n=2$. When the conics of $(x')$ form a pencil, they may be transformed into a line pencil by quadric inversion, by which also the lines of $(x')$ are transformed into conics with three basis points. These cases are then birationally equivalent to particular cases of Class 2. All the above cases would be independent types of Class 1 if they were not forced, as it were, into special cases of types of Class 2 by the fact that the lines or conics of $(x')$ can have but two homogeneous parameters because these parameters represent a pencil in $(x)$. If in Type V the parameters of the line and conic are interchanged, the cubics of $(x)$, since their parameters are now lines of $(x')$, can have but three homogeneous parameters and therefore have two basis points besides those required by the system in $(x)$. By quadric inversion these cubics go into conics through the double point and the two added basis points. The conics of $(x)$ through the two required basis points are invariant. Then the conics, transforms of the cubics, and the invariant conics have one point in common, so this case reduces to a special case of Type III.

In Class 2 if the system in $(x)$ consists of lines and cubics and the lines are forced into a pencil by having the line pencil of $(x')$ for parameters, we have a special case of Type VII for $m=3$. When the system of $(x)$ is composed of conics with a common basis point, the conics having the line pencil of $(x')$ as parameters form a pencil which goes into a pencil of lines by quadric inversion. By the same process the conics in the other equation are transformed into cubics none of whose basis points need be at the vertex of the line pencil. This case, then, is birationally equivalent to Type VII for $m=3$. When the system in $(x)$ consists of cubics with a double and a simple point and conics through these points, if the conics form a pencil they correspond to a line pencil; the cubics are invariant under quadric inversion

so this also reduces to a special case of Type VII. When the parameters are interchanged the cubic pencil, since it contains a double point is transformed into a pencil of conics and thence into a pencil of lines. The conics given by the other equation are transformed into cubics with three basis points, none of which are at the vertex of the line pencil. This is also a special case of Type VII.

In Class 3 when the conics of $(x')$ form a pencil in one equation we have a particular case of the type of Class 2 having the same system of curves in $(x)$. When the system in $(x)$ consists of cubics with a double and a simple point and conics through them, the cubics are given two more basis points because the conics of $(x')$ allow but four homogeneous parameters in the equation. The cubics then reduce to conics through two points one of which is the double point, and as the conics given by the other equation may remain invariant, this case is birationally equivalent to a special case of Type IX.

If, in any combination of the curve systems used in the twelve types, the associated parameters so restrict each other that the curves of each of the two defining equations become pencils in their respective planes, a (2, 3) compound involution is established.

29. *Reduction of All Other Cases.*—Since any combination of the foregoing curve systems of $(x)$ and $(x')$ that has at least three parameters in one of its defining equations is birationally equivalent to some one of the twelve independent types, in order to prove the classification complete it remains only to show that any curve system having two [three] variable intersections is birationally equivalent to some one of the above curve system of $(x')$ $[(x)]$. The proof of this is the same as that for the curve systems of the simple planes of the (1, 2) and (1, 3) point correspondences. From the inequalities connecting intersections of curves of such systems, it follows by reasoning exactly like that used by Bertini* that any given non-involutorial curve system intersecting in two [three] points can be reduced by a series of quadratic transformations to one of the preceding systems of $(x')$ $[(x)]$. Any algebraic method of establishing a general (2, 3) point correspondence between two planes is therefore birationally equivalent to some one of the given twelve independent types.

CORNELL UNIVERSITY, 1917.

* E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano," *Annali di Matematica*, Ser. 2, Vol. VIII (1877), pp. 244—286.

# *The Classification of Plane Involutions of Order (3).*

By Anna Mayme Howe.

## *Introduction and General Discussion.*

1. The purpose of this paper is to discuss all the different algebraic (1, 3) point correspondences between two planes. Such a correspondence is established between two planes $(x)$ and $(y)$ when a relation exists such that to any point in $(x)$ corresponds a point $(y)$, but to any point in $(y)$ corresponds three points $(x)$.

The images of lines on either plane are curves of order $m$ in the other. The simple plane $(x)$ contains curves which are nets, images of lines in $(y)$.

These curves have $n$ variable intersections which are the $n$ points of $(x)$ corresponding to a point $(y)$ determined by the intersection of two lines whose images are the two curves; the remaining $m-n$ intersections are fixed and are common to all the image curves of the net. If two of the $n$ points coincide, two curves of the net are tangent at the point. The locus of all these points of tangency is the curve of coincidences. We shall designate it by $K$. It is always a component of the Jacobian of the net.

There exists in the $(y)$ plane a curve of branch points whose points are in (1, 1) correspondence with those of $K$. This curve of branch points, denoted by $L$, is a fixed curve, every point of which has at least two of its $n$ image points coinciding in a definite direction at a point on the coincidence curve. In case $m>2$ the remaining non-coincident image points describe another locus, the residual image of $L$. This locus will be denoted by $J'$. The points at which more than two of the $n$ images of a point in $(y)$ coincide are intersections of the residual image with the curve of coincidences.

The first (1, 2) transformations discussed were those by Geiser.*

In this discussion a cubic surface is projected on a double plane from a point on it, and also mapped on a single plane by means of two non-intersecting lines on the surface.

In 1871 Clebsch* devised a (1, 2) point correspondence between two planes depending on the bisection of Abelian functions connected with plane

* "Ueber zwei geometresche Probleme," *Crelle's Journal für die Mathematik*, Vol. LXVII (1867), pp. 78–89.

conics, the plane of the conic being so related to a surface that to a point on the surface corresponds one point in the plane, but to a point in the plane corresponds two points on the surface.

The next step was made by de Paolis† who worked out the general theory of (1, 2) point correspondence between two planes as a generalization of the rational transformation of Cremona.

This is the first published treatment of multiple correspondence by means of an algebraic method.

The classification of all possible birational (involutorial) transformations associated with the (1, 2) types was completed by Bertini.‡

Also F. M. Morgan.§ In 1884 Chizzoni‖ developed the general theory of (1, $n$) point correspondence by a plane involution of order $n$ in which he considered a given plane as containing $\infty^2$ series of groups of $n$ points such that each group is fully determined by any one of the points.

Other writers that have made noteworthy contributions to the theory of (1, $n$) point correspondence between two planes are G. Castelnuovo,¶ Miss Charlotte A. Scott,** and Amerigo Bottari.††

All of these treat the subject for a general $n$ except Miss Scott who, in addition, works out several examples for special values of $n$ greater than 2.

### *Enumeration of Independent Types.*

2. In the theory of (1, 3) point correspondence between two planes there are five types, no one of which can be derived from another by means of a

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* "Ueber den Zusammenhang einer Classe von Flächenabbildungen mit der Zweitheilung der Abel'schen Functionen," *Mathematische Annalen*, Vol. III (1871), pp. 45–75.

†(1) "Le transformazioni doppia."

(2) "La transformazione piano doppia di secondo ordine, e le suo applicazione alla geometria non euclidea."

(3) "La transformazione piano doppia di terzo ordine prime genere, e la sua applicazione a curve del quarto ordine," *Atti della R. Accademia dei Lincei*, Series 3, Vol. XII (1877).

‡ "Ricerche sulle transformazioni univoche involutorie nel piano," *Annali di Matematica*, Series 2, Vol. VIII (1877), pp. 244–286.

§ "Involutorial Transformations," AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1913), pp. 79–104.

‖ "Sopra le involuzioni piani," *Atti della R. Accademia dei Lencei*, Series 3, T. 19 (1884), pp. 301–371.

¶ "Sulla razionalita delle involuzioni piane," *Mathematische Annalen*, Vol. XLIV (1894), pp. 125–155.

** "Studies in Transformation of Plane Algebraic Curves," *Quarterly Journal of Mathematics*, Vol. XXIX (1899), pp. 329–381, and *ibid.*, Vol. XXXII (1901), pp. 209–239.

†† "Sulla razionaliti dei piani multipli $\{xy\sqrt[n]{F(x, y)}\}$," *Giornale di Matematiche*, Vol. XLI (1903), pp. 285–320, and also "Sulla razionaliti dei piani multipli $\{x, y\sqrt[n]{F(x, y)}\}$," *Annali di Matematica*, Series 3, Vol. II (1899), pp. 277–296.

birational transformation. For each type there exists in one plane a point determined by two straight lines, and in the other three image points. The five distinct types are defined in the following ways:

(1) a. A field of straight lines, $\Sigma y_i x_i = 0$.

b. A net of cubic curves $\phi_i = 0$ without basis points, $\Sigma y_i \phi_i = 0$.

(2) a. A pencil of lines, $y_1 x_1 + y_2 x_2 = 0$.

b. A net of curves of order higher than 3 and having a fixed point of multiplicity $n-3$ at the vertex of the pencil of lines $y_1 x_1 + y_2 x_2 = 0$. This is associated with Jonquières' configuration in (1, 1) and (1, 2) transformations. A discussion of the (1, 1) type has been presented by P. P. Boyd.*

(3) a. A net of conics through one fixed point.

b. A net of conics through the same fixed point,

$$\Sigma y_i u_i = 0, \quad \Sigma y_i v_i = 0,$$

where $u_i = 0$, $v_i = 0$ are general conics through a fixed point.

(4) a. A net of cubic curves passing through six fixed points.

b. A net of cubic curves passing through the same fixed points,

$$\Sigma y_i u_i = 0, \quad \Sigma y_i v_i = 0,$$

where $u_i = 0$, $v_i = 0$ now represent cubic curves through six basis points.

(5) a. A pencil of cubic curves through nine fixed points.

b. A net of curves of order 9 passing triply through eight of the nine fixed points,

$$y_1 \psi_1 + y_2 \phi_2 = 0, \quad \Sigma y_i \psi_i = 0.$$

If $u_i$ is of order $m_1$ and $v_i$ is of order $m_2$, the transformation curves are of order $m_1 + m_2 = m$. To the line $y_i = 0$ correspond curves $\phi_i(x) = 0$ of order $m$; to the lines $x_i = 0$ correspond curves $\psi_i(y) = 0$ of order $m$.

The fixed points common to all curves of the defining system $u_i = 0$, $v_i = 0$ are fundamental. To each fundamental point $F$ corresponds a curve $f(y)$ whose order is equal to the multiplicity of the point on the curves $\phi_i(x) = 0$, and to each direction through the fundamental multiple point corresponds a point on the image fundamental curve $f(y)$. The complete image of $f(y)$ is composite. The components are the original multiple point and a residual curve.

Every branch of this residual curve has contact with some branch of $K$ through this point. To each of these simple intersections on $L$ is related a

* "On the Perspective Jonquières Involutions Associated with the (2, 1) Ternary Correspondence." AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 290–324.

direction of $J'$ through the original point. Aside from these intersections $J'$ has only points of contact in common with the residual curve and these correspond to the points of tangency of $L$ with $f(y)$. See F. Chizzoni.*

If two curves of the defining system have contact at a given point, then one curve of the net has a double point at the point. Hence the locus of all the double points of curves of the net is the curve $K$.

The order of the Jacobian is $3(m-1)$. The multiplicity of any fundamental point $P_i$ on the $\phi_i=0$ which lies on the Jacobian is $3i-1$.

The equation of $L$ may be obtained by eliminating $(x_i)$ from the equation for $K$ and the equations of transformation $y_i=\phi_i(x)$.

The order of the curve $L$ is the number of intersections aside from those at the fundamental points of $K$ with the image of a straight line $(y)$. This has been expressed by Zeuthen† in the formula $2(k+p-1)$ where $k$ is the multiplicity of the transformation and $p$ is the genus of the curves of transformation, images of the straight lines in $(y)$.

The complete image in $(x)$ of $L$ is a composite curve which has for its order the product of the orders of $L$ and the transformation $\phi$. The image contains $K$ twice and the residual curve $J'$, and also contains fundamental curves, when $L$ passes through their corresponding fundamental points.

## *Type I.*

3. The defining equations for this type are

$$x_1y_1+x_2y_2+x_3y_3=0, \quad u_1y_1+u_2y_2+u_3y_3=0,$$

where $u_i=0$ are cubic curves having no fixed point in common. On solving for $y_i$ we obtain

$$\rho y_1=u_3x_2-u_2x_3=\phi_1(x), \quad \rho y_2=u_3x_1-u_1x_3=\phi_2(x), \quad \rho y_3=u_1x_2-u_2x_1=\phi_3(x).$$

To obtain the points common to the three curves $\phi_i=0$ we write the equations in the form $\frac{u_2}{x_2}=\frac{u_3}{x_3}=\frac{u_1}{x_1}$. Of the sixteen intersections of any two curves as $\phi=0$, $\phi_3=0$ we exclude those which make $u_1=0$, $x_1=0$ as these points do not lie on $\phi_1=0$. Hence we find $16-3=13$ points common to all the curves of the net. These are simple fundamental points.

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* "Sopra le involuzioni piane," *Lincei Memoire*, Series 3, Vol. XIX (1884), p. 301.

† "Nouvelle démonstration de théorèmes sur les séries de points correspondants sur deux courbes," *Mathematische Annalen*, Vol. III (1871), p. 150.

The number of conditions required to determine a quartic curve is fourteen. Since there are thirteen fixed points on each quartic curve, any one of the three variable points uniquely fixes the other two.

The genus of the curves $\phi_i=0$ is 3 since they have no double points. The image curve is a rational quartic curve and has a triple point at $(a_1, a_2, a_3)$. Two of these quartic curves intersect in sixteen points, each of which has three image points in $(x)$. The complete image of this quartic curve is a curve of order 16. This curve is composed of the original line and a curve of order 15. Two of these curves intersect in two hundred and twenty-five points. But if each complete curve passes through each of the thirteen fundamental points four times, aside from these intersections there are $225-4\cdot 4\cdot 13=17$ points. But the two straight lines intersect in one image point, and each line cuts the curve of order 15 in fifteen points. The forty-eight image points are made up of seventeen points on the two curves of order 15, not at the thirteen fundamental points; one point of intersection of the two straight lines and fifteen points on each line through the curve of order 15 with which it is related.

The curve $K$ is of order 9 with double points at the thirteen basis points. Hence it is of genus 15. The number of variable intersections of $K$ with the image in $(x)$ of a straight line in $(y)$ determines the order of the image of $K$. The curve $L$ is, then, of order 10. $9\cdot 4-2\cdot 13=10$. It is also of genus 15, hence it has twenty-one double points or their equivalents. The curve $J'$ is of the order 22 and genus 15, hence has one hundred and ninety-five double points or their equivalents.

The number of intersections of $K$ with $J'$ aside from those at the fixed points are forty-two, or twenty-one contacts, $9\cdot 22-13\cdot 2\cdot 6$. But this is the number of double points of $L$ to be accounted for.

It will be shown that the forty-two intersections of $K$ with $J'$ are twenty-one contacts, images of cusps on $L$. The equation of $L$ is the condition that a line in $(x)$ touches its associated cubic, hence it is the discriminant of a binary cubic equated to zero. These values that make the binary cubic in $x_2$, $x_3$ a perfect cube correspond to cusps on $L$ since the form of the discriminant is $4G^2+H^3$. Any point on $L$ that makes $H=0$ and $G=0$ is such that all three images of the point are coincident. Hence in $(x)$ $K$ and $J'$ have the image point in common.

The cuspidal tangent to $L$ corresponds to the particular quartic of the net which has a node at the corresponding point on $K$.

*Fundamental System.*

4. Each of the thirteen points of the net of quartics when substituted in the defining equations, makes the two equations identical, hence they are all fundamental points, and their images are straight lines in $(y)$. There are no fundamental points in $(y)$, hence no fundamental curves in $(x)$.

The complete image of each of these lines is a curve in $(x)$ of order 4 which has a double point at the original point. For, let $\bar{x}_1y_1+\bar{x}_2y_2+\bar{x}_3y_3=0$ be the equation of the fundamental line in $(y)$ which is the image of the fundamental point $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ which is any of the thirteen points (basis) of the net of quartics.

The image of this in $(x)$ is obtained by making the transformation according to the equations already obtained and we have

$$\bar{x}_1(u_3x_2-u_2x_3)+\bar{x}_2(u_1x_3-u_3x_1)+\bar{x}_3(u_2x_1-u_1x_2)=0.$$

Differentiating this with regard to $x_1$ we have,

$$\bar{x}_1\left(x_2\frac{du_3}{dx_1}-x_3\frac{du_2}{dx_1}\right)+\bar{x}_2\left[x_3\frac{du_1}{dx_1}-\left(u_3+x_1\frac{du_3}{dx_1}\right)\right]+\bar{x}_3\left(u_2+x_1\frac{du_2}{dx_1}-x_2\frac{du_1}{dx_1}\right)=0,$$

substituting in this the coordinates of the point $(x_1, x_2, x_3)$, we have the form written as the expression

$$\bar{x}_1\left(\bar{x}_2\frac{d\bar{u}}{dx_1}-x_3\frac{d\bar{u}_2}{dx_1}\right)+x_2\left(\bar{x}_3\frac{d\bar{u}_1}{dx_1}\bar{u}_3-\bar{x}_1\frac{du_3}{dx_1}\right)+x_3\left(\bar{u}_2+x_1\frac{d\bar{u}_2}{dx_1}-x_2\frac{d\bar{u}_1}{dx_1}\right)=0.$$

But the only terms which do not vanish are $\bar{u}_2\bar{x}_3-\bar{u}_3\bar{x}_2=0$, which equation is satisfied since $(\bar{x})$ is on $u_2x_3-u_3x_2=0$. This quartic curve has one double point and twelve simple fundamental points. The intersections of $K$ and $J'$ with this image curve aside from those at the fundamental points are eight $(9\cdot4-2\cdot2-2\cdot12=8)$ for $K$, and four for $J'$ $(4\cdot22-2\cdot6-6\cdot12=4)$.

At the common double point $K$ touches each branch of the quartic. This is shown as follows: When two of the image points coincide, two or more of the curves are tangent at the basis points of the net and one curve has a double point. If the Jacobian has a double point at a basis point, it is tangent to each branch of the nodal curve of the net at this point.

The Jacobian can then be formed from a net of curves, one of which has at a fundamental point $(\bar{x}_1)=(0, 0, 1)$, a double point. These curves are

$$\psi_1=(x_1x_2)^2x_3^2+(x_1x_2)^3x_3+(x_1x_2)^4=0,$$
$$\psi_2=u_1(x_1x_2)x_3^3+u_2(x_1x_2)x_3^2+u_3(x_1x_2)x_3+u_4(x_1x_2)=0,$$
$$\psi_3=v_1(x_1x_2)x_3^3+v_2(x_1x_2)x_3^2+v_3(x_1x_2)x_3+v_4(x_1x_2)=0.$$

The Jacobian is of the form

$$\begin{vmatrix} x_1x_2^2x_3^2+- & x_2(x_1)_2x_3^2\ +-- & 2(x_1x_2)_2x_3+(x_1x_2)x_3\ +-- \\ u_1(x_1x_2)x_3^2+- & u_2(x_1x_2)x_3^2\ +-- & 3u(x_1x_2)x_3+u_1(x_1x_2)_3+-- \\ v_1(x_1x_2)x_3^2+- & v_2(x_1x_2)^2x_3^2+-- & 3v(x_1x_2)_2x_3+v_1(x_1x_2)_3+-- \end{vmatrix}.$$

By collecting the coefficients of $x_3$ the expanded form is found to be

$$3x_2u_2v-3x_2v_2u-3u_1x_1v+2u_1v_2x_1x_2+3x_1v_1u-2u_2v_1x_1x_2=0,$$

or

$$3v(x_2u_2-u_1x_1)-3u(x_2v_2-x_1v_1)+2x_1x_2(u_1v_2-u_2v_1)=0.$$

But

$$u=ax_1+bx_2,\quad \frac{du}{dx_1}=a=u_1,\quad \frac{du}{dx_2}=b=u_2,$$

and

$$v=cx_1+dx_2,\quad \frac{dv}{dx_1}=c=v_1,\quad \frac{dv}{dx_2}=d=v_2.$$

The Jacobian becomes

$$3(cx_1+dx_2)(bx_2-ax_1)-3(ax_1+bx_2)(dx_2-cx_1)+2x_1x_2(ad-bc)=0,$$

or

$$3[bcx_1x_2+bdx_2^2-acx_1^2-adx_1x_2-adx_1x_2-bdx_2^2+acx_1^2+bcx_1x_2 +2x_1x_2(ad-bc)]=0,$$

or finally

$$2x_1x_2(bc-ad)=0.$$

This is the coefficient of the highest power which does not vanish in the equations of the tangents to the curves of the net passing through $(x_1)$ and is also the coefficient of highest power of $x_3$ in the equation of the Jacobian. It is the form of the tangent to the Jacobian also, and we have proved that the nodal $\phi$ has the same tangent as the Jacobian at the double point. These are $x_1=0$ and $x_2=0$.

Each direction of $J'$ through the point corresponds to one point of intersection of $L$ with the fundamental line. The four remaining points of intersection on $J'$ are two contacts, the images of the two points of contact of $L$ with the fundamental line.

The line in $(y)$ is bitangent to $L$. The six simple intersections are images of the six directions on $J'$ at the fundamental point.

*Type II.*

5. The defining equations for Type II are

$$x_1y_1+x_2y_2=0, \tag{1}$$

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{2}$$

where $u_i=0$ represent curves of order $n>3$ having an $n-3$-fold point in common. Let this point be $P(x)=(0,0,1)$. Then

$$\rho y_1=u_3x_2=\phi_1(x),\quad \rho y_2=-u_3x_1=\phi_2(x),\quad \rho y_3=u_2x_1-u_1x_2=\phi_3(x).$$

These curves $\phi_i=0$ have an $(n-2)$-fold point at $P(x)$. The equation of the image of a general line in $(x)$ has the form

$$\begin{aligned}(a)\quad & y_1\{(a_1y_2-a_2y_1)^3[f_1(a_3-y_1y_{2_{n-3}})]+(a_1y_2-a_2y_1)^2[f_1(a_3-y_1y_{2_{n-2}})]\\ &+(a_1y_2-a_2y_1)[f_1(a_3-y_1y_{2_{n-1}})]+f_1(a_3-y_1y_{2_n})\}\\ &+y_2\{(--)\}+y_3\{(--)\}=0.\end{aligned}$$

Let

$$\begin{aligned} f_i(a_3,-y_1y_{2_{n-3}})&=v_i, & f_i(a_3,-y_1y_{2_{n-1}})&=s_i,\\ f_i(a_3,-y_1y_{2_{n-2}})&=w_i, & f_i(a_3,-y_1y_{2_n})&=t_i,\end{aligned}$$

the equation of the curve is then

$$\begin{aligned}(b)\quad & (a_1y_2-a_2y_1)^3(y_1v_1+y_2v_2+y_3v_3)+(a_1y_2-a_2y_1)^2(y_1w_1+y_2w_2+y_3w_3)\\ &+(a_1y_2-a_2y_1)(y_1s_1+y_2s_2+y_3s_3)+(y_1t_1+y_2t_3+y_3t_3)=0.\end{aligned}$$

This curve is of order $n-3+3+1=n+1$ having at $P(y)$ an $n$-fold point.

Aside from the intersections at $P(y)$ two curves of order $n+1$ will have $(n+1)^2-(n)^2=2n+1$ points in common, of which one is variable.

The image of a line through $P(x)$ is found by dividing the above equation (a) by $a_3^{n-3}$ and then putting $a_3=0$. All terms containing $a_3$ to a power greater than $n-3$ will vanish. The remaining terms have the form

$$(a_1y_2-a_2y_1)^3[y_1f(y_1y_{2_{n-3}})+y_2f_2(y_1y_{2_{n-3}})+y_3f_3(y_1y_{2_{n-3}})]=0,$$

which consists of a line $a_1y_2-a_2y_1=0$ through $P(y)$ taken three times, and a curve $\Sigma y_if_i(y_1y_{2_{n-3}})=0$ of order $n-2$, having at $P(y)$ an $n-3$-fold point, and is independent of the coefficients $a_i$. The line is the image of the given line, and the curve is the image of the point $P(x)$.

### *The Fundamental System.*

6. The $6(n-1)$ basis points are all fundamental; their images in $(y)$ are straight lines through $P(y)$.

The complete image in $(x)$ of each of these fundamental lines in $(y)$ is a curve of order $n+1$ having an $(n-2)$-fold point at $P(x)$. But, since the line in $(y)$ passes through $P(y)$, the image of $P(y)$ must be deducted. It is a curve $C_1$ passing through $P(x)$, and the one point of the $6(n-1)$ basis points of which the line in $(y)$ is an image. The proper image in $(x)$ of the line in $(y)$ is, then, of order $n$ having an $n-3$-fold point at $P(x)$ and passing once through each of the $6(n-1)$ points.

The point $P(x)$ of order $n-2$ on the transformation $\phi_i=0$ has a curve of order $n-2$ as its image in $(y)$, with a point of multiplicity $n-3$ at $P(y)$. The complete image of this curve in $(x)$ is of order $(n+1)(n-2)-n(n-3)=2(n-1)$, having at $P(x)$ a multiplicity $2n-4$ and passing once through each of the $6(n-1)$ points.

The curve $K$ is of order $3n$, having at $P(x)$ a point of multiplicity $n-2+n-2+n-3=2n-7$. But $u_3$ is a factor of each term, so that the curve $K$ is of order $2n$, having a $2n-4$-fold point. It also passes simply through the $6(n-1)$ simple basis points. The genus of $K$ is $6n-9$.

The order of $L$ is $2n(n+1)-(n-2)(2n-4)-6n-6=4n-2$, and its genus is $6n-9$. The curve has at $P(y)$ a point of multiplicity $4n-6$ and a double point at each of the $6(n-1)$ basis points. The curves $K$, $J'$ have $6(n-1)$ contacts and $L$ has $6(n-1)$ cusps.

Let $P_i$ denote a simple fundamental point in $(x)$; $p_i$ denote the image in $(y)$ of $P_i$, and let $r_i$ denote the residual image curve in $(x)$ of $p_i$.

Thus $J'$ has a double point at $P_i$ and one contact not at the fundamental point corresponding to the tangency of $L$ with $p_i$.

We denote the point in $(x)$ by $Q$, its image curve in $(y)$ by $q$ and the residual curve in $(x)$ by $s$.

$K$ passes through $Q$ $2n-4$ times and is tangent to each of the $n-2$ branches of $\phi_i$. There are $2n-4$ points of tangency of $L$ with $q$, leaving $4n-6$ simple intersections. These simple intersections correspond to the directions of $J'$ through $Q$.

The intersections of $K$ with $s$, not at the fundamental points are shown to be $4n-6$. These intersections correspond to the simple intersections of $L$ with $q$. Aside from the fundamental points, $J'$ has $4n-8$ intersections or $2n-4$ contacts with $s$ which correspond to the points of contact of $L$ with $q$.

*Type III.*

7. The defining equations for Type III are

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_1+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$, $v_i=0$ are general conics through $P=(0,0,1)$.

*The Fundamental System.*

8. The image of a point $P(x)$ is a curve $p(y)$ of order 2 in $(y)$. The complete image in $(x)$ of $p(y)$ is a curve of order 8 having a five-fold point at $P(x)$ and double points at each of the nine simple basis points.

The image of each of the nine simple basis points in $(y)$ is a line whose complete image in $(x)$ is a curve of order 4 belonging to the net having a double point at $P(x)$ and also at the given basis point.

$K$ is of order 9, genus 9. There are no fundamental points in the $(y)$ plane. The complete image of $K$ consists of the conic $p(y)$ taken five times; images of the double point $P(x)$ through which $K$ passes five times; nine straight lines each counted twice, images of the nine simple basis points, through which $K$ passes twice, and also the curve of branch points $L$.

$L$ is of order 8, genus 9; it has twelve cusps. $J'$ has a six-fold point at $P(x)$; $J'$ and $K$ have twelve points of contact.

We shall denote one of the nine basis points of the net by $Q_i$, the straight line in $(y)$ which is its image by $q_i$ and the residual image curve in $(x)$ by $s$. We shall denote the image of the fundamental point $P(x)$ of multiplicity 2 on $\phi_i$ by $p$ and its residual curve in $(x)$ by $r$.

As in Type I, $L$ is tangent to $q$ as many times as $K$ passes through $Q$. Since $K$ has a double point at each of the nine simple basis points, $q$ is tangent to $L$ twice. $L$ and $q$ have $1 \cdot 8 - 2 \cdot 2 = 4$ simple intersections, each of which corresponds to a direction of $J'$ through $Q$.

$K$ has four simple intersections with $s$ not at fundamental points, the images of the four simple intersections of $L$ with $q$.

$J'$ has four simple intersections or two contacts with $s$ corresponding to the two contacts of $L$ with $q$.

Corresponding to the five contacts of $K$ with $r$ at $P(x)$ are five contacts of $L$ with $p$, which leaves six simple intersections of $p$ with $L$. These correspond to the directions of $J'$ through $P(x)$.

$K$ has six simple intersections with $r$ besides those at the fundamental points. These are images of the simple intersections of $L$ with $p$.

$J'$ has ten simple intersections or five contacts with $r$ not at fundamental points. These correspond to the five contacts of $L$ with $p$.

In the general case of two nets of conics through a common point we have the configuration of quartic curves having a double point and nine simple points. As a subcase is included a system of conics having a simple point in common and no basis points. This system is a linear combination of conics forming a net.

*Type IV.*

9. The defining equations for Type IV are

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_1+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$, $v_i=0$ are cubic curves passing through six fixed points. We shall denote these points by $P_i$.

The equations of transformation are

$$\rho y_1=u_2v_3-u_3v_2=\phi_1(x), \text{ etc.}$$

The curves $\phi_i$ have a double point at each of the six $P_i$. They also intersect in $6\cdot 6-6\cdot 2\cdot 2-3=9$ simple points of the system. We shall denote these basis points by $Q_i$. Each $\phi_i$ is of genus 4. The image in $(y)$ of a line in $(x)$ is a sextic curve.

*The Fundamental System.*

10. Each $P_i$ of the net as well as each $Q_i$ when substituted in the equations (1) and (2) make them identical. Hence they are fundamental points.

The image of each $P_i$ in $(y)$ is a conic which we shall denote by $p_i$. The residual image of $p_i(y)$ in $(x)$, *i. e.*, $r_i$ is a curve of order 12 having at the given $P_i$ a five-fold point. The residual curve $r_i$ has a double point at $Q_i$.

The image of each $Q_i$ is a straight line $q_i$ in $(y)$ and a residual curve $s_i$ in $(x)$ of order 6 belonging to the net. At the given $Q_i$ the curve $r_i$ has a double point and a simple point at each of the remaining eight $Q_i$. The curve $r_i$ has a double point at each $P_i$.

$K$ is of order 15, genus 22; $L$ is of order 12, and has thirty-three cusps. Aside from $L$ the complete image of $K$ is composed of nine straight lines taken twice, images of the nine basis points through which $K$ passes twice and six conics taken five times, images of the six $P_i$ through each of which $K$ passes five times.

The complete image of $L$ in $(x)$ is a curve of order 72 of which $K$ is a double component. The residual image $J'$ is of order 42. The complete multiplicity of each $P_i$ on $K^2J'$ is 24 from which a five-fold point taken twice must be deducted for $K^2$. Hence $P_i$ is of multiplicity 14 on $J'$. The total number of intersections at each $Q_i$ is twelve, of which four are on $K^2$. $J'$ has at each $Q_i$ an eight-fold point. The genus of $J'$ is 22. $K$ and $J'$ intersect in $42\cdot 15-6\cdot 14\cdot 5-9\cdot 2\cdot 8=66$ points not at fundamental points which are thirty-three contacts corresponding to the thirty-three cusps on $L$.

Each $q_i$ meets $L$ in twelve points. As Type I, $L$ is tangent to $q_i$ as many times as $K$ is multiple at $Q_i$. Hence there are two contacts of $L$ with $q_i$ and eight simple intersections each of which corresponds to a particular direction of $J'$ through $Q_i$.

$K$ intersects $r_i$ in eight points not at fundamental points which correspond to the eight simple points of intersection of $L$ with $q_i$.

$J'$ has four simple intersections or two contacts with $r_i$ which are images of the two contacts of $L$ with $q_i$.

In the same way $L$ meets $p_i$ in twenty-four points of which five are contacts and fourteen are simple intersections. The points of tangency correspond to the contacts of $K$ with $s_i$ at $P_i$. Each of the simple intersections corresponds to a particular direction of $J'$ through $P_i$.

There are fourteen intersections of $K$ with $s_i$ not at fundamental points, images of the fourteen intersections of $L$ with $p_i$.

There are ten intersections of $J'$ with $s_i$, not at fundamental points, which are five contacts corresponding to the five contacts of $L$ with $p_i$.

### *Type V.*

11. The defining equations for Type V are

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_1+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$ represent cubic curves passing through nine fixed points, and $v_i=0$ represent curves of order 9, having a triple point at eight of the nine fixed points of $u_i=0$. The equations of transformation are

$$\rho y_1=u_2v_3=\phi_1(x), \quad \rho y_2=-u_1v_3=\phi_2(x), \quad \rho y_3=u_2v_1-u_1v_2=\phi_3(x).$$

There are eight points $P_i$ of multiplicity four on each $\phi_i$, and thirteen simple basis points $Q_i$. The fundamental curve $v_3=0$ has a three-fold point at each $P_i$ and passes simply through twelve of the thirteen $Q_i$. $9\cdot 12-8\cdot 3\cdot 4=12$. We shall denote by $Q_m$ the basis point through which $v_3=0$ does not pass.

### *The Fundamental System.*

12. The image in $(y)$ of each fundamental point $Q_i$ is a straight line $q_i$ passing through $P(y)$. The residual image in $(x)$ is a composite curve of order 12 having eight points of multiplicity, four at $P(x)$, passing simply through eleven of the $Q_i$, and doubly through the given $Q_i$. One component

of this curve is $v_3=0$. The remaining image $r_i$ is a curve of order 3 having a simple point at each $P_i$ and also at the given $Q_i$ and $Q_m$.

The image in $(y)$ of $Q_m$ is a straight line $q_m$ which does not pass through $P(y)$. The residual image $r_m$ in $(x)$ is a curve of order 12 having a four-fold point at each $P_i$, a simple point at each $Q_i$, and a double point at $Q_m$.

The image of each $P_i$ is a curve $p_i$ of order 4 having a three-fold point at $P(y)$. The residual image in $(x)$ is a composite curve of order 48, having a sixteen-fold point at seven $P_i$ and a seventeen-fold point at the given $P_i$. The multiplicity at $Q_i$ and $Q_m$ is 4. One component is the image of $P(y)$ taken as many times as $p_i$ passes through $P(y)$. The remaining component which is the curve $s_i$ of order 21, having a multiplicity of 7 at seven of the $P_i$ and of 8 at the given $P_i$. At each $Q_i$ there is a simple point, but at $Q_m$ there is a four-fold point.

The Jacobian is of order 33. This is a composite curve, one component being $v_3=0$. Hence the curve of coincidences $K$ is of order 24, having eight-fold points at $P_i$, simple points at $Q_i$ and a double point at $Q_m$. The genus of $K$ is 28. The order of $L$ is 18. Corresponding to each intersection of $K$ with $v_3=0$, not at the fundamental points, is a branch of $L$ through $P(y)$. $24\cdot 9-8\cdot 8\cdot 3-12\cdot 2=12$. Hence $L$ has a twelve-fold point at $P(y)$. The genus of $L$ is 28, and there are forty-two double points, or their equivalents, to be accounted for. The curve $J'$ is of order 60.

The total multiplicity at $P_i$ is 72, at $Q_i$ the total multiplicity is 18, and at $Q_m$ it is 18. Deducting the multiplicities at these points on $K^2$ and $v_3^{12}=0$ there remains a twenty-fold point at $P_i$, a four-fold point at $Q_i$, and a fourteen-fold at $Q_m$ on $J'$. $K$ and $J'$ have, aside from the fundamental points, forty-two contacts which correspond to the forty-two cusps on $L$.

As in Type I, $L$ is cut by $q_m$ in eighteen points, two of which are contacts corresponding to the tangency of $K$ with $r_m$ at $Q_m$. The fourteen simple intersections of $K$ with $r_m$, not at fundamental points, are images of the fourteen simple intersections of $q_m$ with $L$.

The four points of intersection of $J'$ with $r_m$, not at fundamental points, are two points of contact, images of the two contacts of $L$ with $r_m$.

$L$ has six points in common with $q_i$, not at $P(y)$, of which one is a point of contact and four are simple intersections. These correspond to the tangency of $K$ with $r_i$ at $Q_i$ and to the directions of $J'$ through $Q_i$.

The intersections of $K$ with $r_i$, not at fundamental points are four, the images of the four intersections of $L$ with $q_i$.

The intersections of $J'$ with $r_i$, not at fundamental points, are two or one point of contact which is the image of the contact of $L$ with $q_i$.

$L$ has thirty-six intersections with $p_i$, not at $P(y)$. Eight of these are contacts and twenty are simple intersections. These correspond to the tangency of $K$ with $s_i$ at $P_i$ and the multiplicity of $J'$ at $P_i$.

There are twenty simple intersections of $K$ with $s_i$, not at fundamental points, which are images of the simple intersections of $L$ with $p_i$.

There are sixteen intersections of $J'$ with $s_i$, not at fundamental points, which are eight contacts, images of eight contacts of $L$ with $p_i$.

There are eight four-fold points and thirteen simple basis points in the general case of a system composed of a net of cubic curves, and a net of curves of order 9. A subcase of this is a linear system of curves of order 9 having eight triple points and six simple basis points. Hence the latter type which is used here is a subcase of the general system composed of the net of cubics and the net of curves of order 9.

*Completeness of Enumeration.*

13. The three images of a point in $(y)$ may be defined as the variable intersections of systems of curves in $(x)$ in many ways. They are, however, reducible to one of the five types described above. For example, the defining equations may be

$$u_1y_1+u_2y_2+u_3y_3=0, \tag{1}$$

$$v_1y_2+v_2y_2+v_3y_3=0, \tag{2}$$

where $u_i=0$ represent cubic curves having a double point at $P(x)$ and a simple point at $Q(x)$, and where $v_i=0$ represent conics passing through $P$ and $Q$. $u_i=0$ have the form $x_3(x_1x_2)_2+x_1(x_1x_2)=0$, and the curves $v_i=0$ have the form $x_3(x_1x_2)+x_1(x_1x_2)=0$. In $\Sigma u_i=0$ there are $\infty^5$ degrees of freedom, and in $\Sigma v_i=0$ there are $\infty^3$ degrees of freedom.

If the $u_i=0$ have the added restriction that they pass simply through a third point $R=(1,0,0)$ through which $v_i=0$ do not pass, the net has $\infty^2$ degrees of freedom. We shall denote the net of these cubic curves through one double and two simple points by $y_1\psi_1+y_2\psi_2+y_3\psi_3=0$. By a quadratic inversion the $\psi_i$ becomes curves of order 6 which contain the image of the double point $P$ and the two simple points $Q$ and $R$. The residual curve is of order 2 which passes once through $P$ but not through $R$ or $Q$.

The conic of the net $v_i=0$ through $P$ and $Q$ become composite curves of order 4 whose residual image is a conic passing through $P$ and $Q$.

The curves of transformation $\rho y_i = \phi_i$ of the system $\Sigma \psi_i y_i = 0$, $\Sigma v_i y_i = 0$ obtained by finding the values of $y_i$ are of order 5 having at $P$, and $Q$ points of multiplicity 3 and 2 and of multiplicity 1 at the nine fixed basis points $T_i$. By a quadratic inversion the $\phi_i$ are of order 10. These are composite curves whose residual images are quartic curves, having one double point and nine simple points. The genus of the quintic is 2 as is also that of the quartic.

The curve of coincidences is of order 11 having at $P$, $Q$, and $T_i$ multiplicities of 7, 4 and 2. By the inversion $K$ is a composite curve of order 22 whose residual image is a curve of order 9 with a point of multiplicity 2 and 5 at $P$ and $T_i$. The genus of $K$ is 9.

Hence the $\psi_i$ reduce to conics through one fixed point, $v_i$ become conics through this same fixed point and a point $M$ through which the first conics do not pass. The transformation $\phi_i$ reduce to quartics having a double point and nine simple points. $K$ reduces to a curve of order 9 and of genus 9 with one five-fold point and nine double points. But this is exactly the configuration for Type III. Thus the system of a net of curves of order 3 having a fixed double point and a fixed simple point through which pass a net of conics reduces by a birational transformation to the system of Type III.

The same method of reduction can be employed in every case. The procedure is exactly that used by Bertini in the paper cited.

### *Cyclic Involutions.*

14. In order that the discussion of the (1, 3) correspondences may be complete it is necessary to consider those cases in which the images of a point $(y)$ are rationally separable such that to a point $P(y)$ correspond three points $P_1(x)$, $P_2(x)$, $P_3(x)$ having the following property. If $P_1$ describe a locus, $P_2$ describes another locus and $P_3$ another locus, all in (1, 1) correspondence. If $P_1$ and $P_2$ describe the same locus, $P_3$ describes the same locus. Hence a birational transformation exists by means of which the three points $P_i$ are permuted among themselves. By this transformation $P_1 \backsim P_2$, $P_2 \backsim P_3$, $P_3 \backsim P_1$. Since the three image points form a group the transformation is cyclic and of order or period 3. Such an involution is called cyclic.

If two of the three image points coincide all three are identical. For the transformation which sends $P_1$ into $P_2$ also sends $P_2$ into $P_3$ and $P_3$ into $P_1$. Hence if $P_1 = P_2$ then $P_1 = P_2 = P_3$. Hence $K$ and $J'$ are identical. There is a birational relation between the curves $K$ and $L$.

Bottari * has discussed the possible forms of all cyclic types by means of projection from configurations in space of three or more dimensions.

For a cyclic transformation of order 3 there are four types among which are included all the Cremona transformations of order 3.

(1) The Jonquières transformation of three points on a fixed line.

(2) The Jonquières transformation of three points on a pencil of lines $p_i$, the lines themselves being permuted by a linear transformation of order 3. In particular $P_1$ on $p_1$ is transformed into $P_2$ on $p_2$, $P_2$ on $p_2$ is transformed into $P_3$ on $p_3$ which in turn becomes the original point. In the same way $P_2$ and $P_3$ on $p_1$ have uniquely associated points on $p_2$ and $p_3$ such that the three points on each $p_i$ form a distinct group.

(3) The three permuted points are the three intersections of two cubic curves through six basis points which form two triads of points in three-fold perspective. The transformation is accomplished by means of a quartic curve through six points. One triad consisting of double points, the other of simple points. Kantor † calls the type $\Delta_3$.

(4) A particular case of Type V, in which the curves of order 9, and a pencil of cubic curves are invariant under a Cremona transformation of period 3 and order 13. This type is called $N_3$ by Kantor (*loc. cit.*, p. 288).

### *Cyclic Type I.*

15. Given a curve $C_n$ with a fixed point of multiplicity $n-3$. The three points in which a line through the fixed point cuts the curve are images of the point of intersection of two lines in $(y)$. The points of each triad are on an invariant line.

The equations of the transformation have the form

$$\rho x_1' = \frac{ax_1+b}{cx_1+d}, \quad \rho x_2' = x_2, \quad \rho x_3' = x_3.$$

It is of period 3 when $ad+bc+a^2+d^2=0$. Equating $T^2$ and $T^{-1}$ we have

$$x^2(acd+cd^2)+x(d^3-abc)-bcd-bd^2 = x^2(-a^2c-bc^2)+x(a^3-bcd)+a^2b+abd+abd.$$

The condition for this equality to exist is found by putting

$$b = \frac{-(a^2+d^2+ad)}{c}.$$

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* "Sulla razionalita dei piani multipli $\{x, y, \sqrt[n]{F(x,y)}\}$." *Annali di Matematica.* Series 3, Vol. II (1899), p. 278.

† "Premiers Fondements pour une Théorie des Transformations Periodiques Univoques," *Memoire, della R. Accademia.* Naples. Series 2, Vol. III, e Vol. IV, 2 (1891), pp. 1-335. See pp. 260-262.

*Locus of Invariant Points.*

16. The invariant points on each line descrive curves that are rationally separable, hence $K$ consists of two distinct rational curves. The points will be invariant when $x_1'=x_1$. Thus, $x_1=\frac{ax_1+b}{cx_1+d}$. The two roots of the equation of $x$ are $x_a=\frac{(a-d)+(a+d)\sqrt{-3}}{2c}$, $x_b=\frac{(a-d)-(a+d)\sqrt{-3}}{2c}$.

A curve of order $n$ having an $n$—3-fold point at $P(x)=(1,0,0)$ has the form

$$x_1^3u_i(x_{2_{n-3}}x_3)+x_1w_i(x_{2_{n-2}}x_3)+x_1w_i(x_{2_{n-1}}x_3)+s_i(x_2x_{3_n})=0.$$

A curve is invariant under $T$ when its equation is of the form

$$(x_1-u_i)(x'_1-u_i)(x''_1-u_i)=k \qquad (1)$$

for the curve $\phi_i$ of the net, wherein

$$x_1'=T(x), \quad x_1''=T^2(x).$$

If the curve (1) is cut by any line which passes through the fixed point $T$ since the line is invariant under the transformation the three points are rationally separable and can be uniquely determined as a cyclic transformation of period 3.

*Quadratic Cyclic Transformations.*

17. The quadratic transformations of period $n$ have been discussed by V. Snyder.* Of these transformations, those that concern us can be transformed into linear perspectivities.

*Cyclic Type II. Non-perspective Jonquières.*

18. In Type II the three images are the variable intersections of two curves $\phi_i$ and $\psi_i$ of degrees $n$ and $m$ having at $P$ points of multiplicity $n$—3 and $m$—3. Each of the three image points lies on a separate line of the pencil through $P_1$, the three lines being permuted by means of the transformation. The transformation has the form $T$ in which $a$, $b$, $c$, $d$ now have the previous relation and the additional restriction that $x_2$ appears only in powers of three in equation (1) of the curve above.

The curves $\phi_i$ and $\psi_i$ are each invariant as a whole under the transformation. Hence, to an arbitrary point $P_1$ there correspond two other points $P_2$

* "Periodic Quadratic Transformations in the Plane," *Annals of Mathematics*, Series 2, Vol. XIII (1912), pp. 140–148.

and $P_8$ which are rationally distinct and lie on the curves. The $k$ points occur in groups of three or $3k'=k$. The equations give us for $y_i$ the values

$$\rho y_1=\phi_2\psi_3-\phi_3\psi_2, \text{ etc.}$$

The images of lines in $(y)$ are curves of order $m+n$, having at each $k'$ point a multiplicity of $r+s$, where $r$ is the multiplicity on $\phi_i$ and $s$ the multiplicity on $\psi_i$. The curves $\phi_i=0$, $\psi_i=0$ are now to be restricted to pass through $k'$ arbitrary points in the plane.

It is a curve $C_{k'}$ of order $r+k$ taken three times, and has a point of such multiplicity $\alpha_i$ as will make the curve rational. The complete image of this curve $C_{k'}$ in $(x)$ is of order $(r+s)(m+n)-\Sigma(m+n)\alpha_i$.

The locus of $K$ consists of the one line $x_3=0$.

### *Cyclic Type III.*

19. Type III consists of the configuration of the three image points in $(x)$ lying on a system of cubic curves with six basis points. In Type V of the general case to which this is related the defining equations are

$$u_1y_1+u_2y_2+u_3y_3=0, \quad v_1y_1+v_2y_2+v_3y_3=0,$$

where $u_i=0$, $v_i=0$ are cubic curves through six fixed points.

To build this type it is necessary to find a birational transformation for the points in $(x)$ such that $T^3=1$.

A transformation composed of the product of two inversions will send a line into a quartic curve through three double points and three fixed simple points. The six basis points are disposed of as a triad of double points and a triad of simple points.

Let $A, B, C$; $\overline{A}, \overline{B}, \overline{C}$ be the two triads of fundamental points. We shall denote as double points $A=(1,0,0)$, $B=(0,1,0)$, $C=(0,0,1)$.

By the transformation $T$ the line $A\overline{A}$ becomes a quartic curve passing through the image of $A$ and of $\overline{A}$. The residual image of $A\overline{A}$ is a line $B\overline{B}$ through two of the remaining four fixed points. By a second transformation $T^2$ the line $B\overline{B}$ has for its residual image the line $C\overline{C}$. Since the transformation is of period 3 $C\overline{C}$ goes into $A\overline{A}$, and it has been shown by Kantor (*loc. cit.*, p. 260), that the two triads of points are in three-fold perspective. We shall denote as simple points $\overline{A}=(a,b,c)$, $\overline{B}=(a,\omega b,\omega^2 c)$, $\overline{C}=(a,\omega^2 b,\omega c)$.

This means that $A\overline{A}$ and the lines into which it is transformed by $T$ and $T^2$ pass through a common point. Also $A\overline{C}$ and its associated lines pass through a common point. We shall denote these points by $P$, $Q$, and $R$.

In order to obtain the relations of the six points in $(x)$ under the transformation, consider three planes $x$, $x'$, $x''$. Under $T$ the six $(x')$ images of the six fundamental points $(x)$ are obtained. These under $T^2$ define six points $(x'')$ and these in turn go into the original $(x)$ points. The $(x')$ and $(x'')$ planes are superposed on the $(x)$ plane to obtain the required relations.

We shall consider the two triads of points in $(x)$ and $(x')$ as

$$(A_2B_2C_2\overline{A}_1\overline{B}_1\overline{C}_1) \quad \text{and} \quad (A_2'B_2'C_2'\overline{A}_1'\overline{B}_1'\overline{C}_1').$$

The points $A_2$, $B_2$, $C_2$ each have conics in $(x')$ and the points $\overline{A}_1$, $\overline{B}_1$, $\overline{C}_1$ have straight lines in $(x')$ for their images. Also the point

$$\begin{aligned} A_2' &\text{ goes into } C_2(ABC\overline{B}\overline{C}), & \overline{A}_1' &\text{ goes into } C_1(BC),\\ B_2' &\quad\text{“}\quad C_2(ABC\overline{A}\overline{C}), & \overline{B}_1' &\quad\text{“}\quad C_1(AC),\\ C_2' &\quad\text{“}\quad C_2(ABC\overline{A}\overline{B}), & \overline{C}_1' &\quad\text{“}\quad C_1(AB). \end{aligned}$$

Since any straight line in $(x')$ goes into a quartic through $(A_2B_2C_2\overline{A}_1\overline{B}_1\overline{C}_1)$, the image of the line $A'\overline{A}'$ is a quartic curve containing the image of $A'$ and $\overline{A}'$. These are the straight line $BC$ and the conic through $ABC\overline{B}\overline{C}$. The residual image is, then, the line $A\overline{A}$. In the same way the images of other lines may be obtained.

Since $CB\equiv x_1=0$, $AC\equiv x_2=0$, $AB\equiv x_3=0$, it follows that

$$\begin{aligned} x_1&= k(bcx_1^2-a^2x_2x_3)(abx_3^2-c^2x_1x_2),\\ x_2&= l(acx_2^2-b^2x_1x_3)(bcx_1^2-a^2x_2x_3),\\ x_3&= m(acx_2^2-b^2x_1x_3)(abx_3^2-c^2x_1x_2). \end{aligned}$$

The points $P$, $Q$, and $R$ are invariant under the transformation, hence the substitution of the coordinates of one of these points in the above equations gives the values of $k$, $l$, and $m$. These are $\omega^2ab$, $\omega bc$ and $ac$, respectively.

The equations of transformation $T^2$ are

$$\begin{aligned} x_1'&= ab(bcx_1^2-a^2x_2x_3)(abx_3^2-c^2x_1x_2),\\ x_2'&= bc(acx_2^2-b^2x_1x_3)(bcx_1^2-a^2x_2x_3),\\ x_3'&= ac(acx_2^2-b^2x_1x_3)(abx_3^2-c^2x_1x_2). \end{aligned}$$

Similarly for $T$ we obtain

$$x_1'=ax_1x_2(abx_3^2-c^2x_1x_2), \quad x_2'=bx_2x_3(bcx_1^2-a^2x_2x_3), \quad x_3'=cx_1x_3(acx_2^2-b^2x_1x_3).$$

The nine points $A$, $B$, $C$, $\overline{A}$, $\overline{B}$, $\overline{C}$, $P$, $Q$, $R$ lie on a cubic curve which is invariant, but since two triads of lines pass through these points, they are the basis points of a pencil of cubic curves. The equations have the form

$$A\overline{A}\cdot B\overline{C}\cdot\overline{B}C+\lambda C\overline{C}\cdot\overline{A}B\cdot A\overline{B}. \tag{1}$$

If any curve of the pencil is invariant under $T$, we must have

$$C\bar{C}\cdot\bar{A}B\cdot\bar{B}A+\bar{\lambda}B\bar{B}\cdot A\bar{C}\cdot C\bar{A}=0, \tag{2}$$

such that (1) and (2) define the same curve. Equation (1) has the form

$$(bx_3-cx_2)(\omega bx_1-ax_2)(\omega cx_1-ax_3)+\lambda(\omega^2bx_1-ax_2)(cx_1-ax_3)(\omega cx_2-bx_3)=0, \tag{3}$$

which by $T$ goes into

$$(\omega^2bx_1-ax_2)(ax_3-cx_1)(bx_3-\omega cx_2)+\bar{\lambda}(ax_3-\omega^2cx_1)(\omega bx_3-cx_2)(bx_1-ax_2)=0. \tag{4}$$

By comparing the coefficients of like terms in equations (3) and (4) we obtain the relation $\bar{\lambda}=\frac{\omega}{\omega+\lambda}$. By the substitution of $\frac{\omega}{\omega+\lambda}$ for $\bar{\lambda}$ in equation (4) we find $\lambda=\omega$ and $\lambda=\omega^2$. The two curves are

$$bc^2x_1^2x_2+\omega ca^2x_2^2x_3+\omega^2ab^2x_1x_3^2=0, \tag{5}$$

$$b^2cx_1^2x_3+\omega^2ac^2x_1x_2^2+\omega a^2bx_2x_3^2=0. \tag{6}$$

The existence of the two cubic curves satisfied by different values of $\lambda$ has been mentioned by H. S. White.*

It is necessary to determine whether either of these cubic curves contains points invariant under the transformation. The point of contact of a line through $A=(0,0,1)$ or $x_1=mx_2$ with (5) gives $x_1=\omega a\sqrt[3]{4}$, $x_2=b\sqrt[3]{4^2}$, $x_3=-2\omega c$. Under $T$ these values become $x_1=(\sqrt[3]{4^2})\omega a$, $x_2=-2\omega b$, $x_3=\sqrt[3]{4}c$. Hence this point is not invariant. The point of contact of $x_1=mx_2$ with (6) gives $x_1=a\sqrt[3]{4^2}$, $x_2=b\omega^2\sqrt[3]{4}$, $x_3=-2c$. Under $T$ this point is invariant, hence the curve (6) is the equation of $K=J'$.

All the cubic curves through $A, B, C, \bar{A}, \bar{B}, \bar{C}$ which contain the three permuted points which are images of a point $(y)$ form a net of equianharmonic curves; they are tangent to one of the sides $x_1=0$, $x_2=0$, $x_3=0$ and do not pass through $P$, $Q$ and $R$.

Since a line in $(y)$ becomes a cubic curve in $(x)$ any one of these equianharmonic curves may be taken for $\phi_i=0$ in the equations of transformation which written in the form of defining equations are $\rho y_i=\phi_i$.

The equation of $L$, the image of $K$, is also a cubic curve. To a line in $(x)$ corresponds a cubic in $(y)$ which has for its complete image in $(x)$ a composite curve made up of the original line and two quartic curves. The line when operated upon by the transformation $T$ becomes one of these quartic curves, and when operated upon by $T^2$ becomes the second quartic which accounts for the curves of order 9 as $C_1+C_4+C_4$.

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* "Plane Cubics and Irrational Covariant Cubics," *Transactions American Mathematical Society*, Vol. I (1900), pp. 170–178.

*Non-perspective Linear Transformations.*

20. It was shown in Art. 17 that quadratic transformations of period 3 can be reduced to the Jonquières type of cyclic involution.

We now consider conics through one fixed point, invariant under the non-perspective cyclic linear transformation of period 3.

If $\rho x_1' = \omega x_1$, $\rho x_2' = \omega^2 x_3$, $\rho x_3' = x_3 (\omega^3 = 1)$, the expression $x_1 x_2 + k x_3^2$ remains invariant, $x_1 x_3 + l x_2^2$ is multiplied by $\omega^2$, and $x_2 x_3 + m x_1^2$ is multiplied by $\omega$ by the transformation. Hence the systems of conics

$$x_1 x_2 y_1 + x_3^2 y_2 = 0, \quad x_2^2 y_3 + x_1 x_3 (y_1 + y_2 + y_3) = 0$$

defines a triad of variable intersections which is invariant under the transformation.

For $K$ we find $x_1 x_2^2 x_3^2 (x_1 x_2 - x_3^2)(x_1 x_3 + x_2^2) = 0$, which is satisfied only by fundamental curves. This is a particular case of Cyclic Type III.

Consider the $\infty^3$ system of cubic curves

$$\alpha_1 x_1^3 + \alpha_2 x_2^3 + \alpha_3 x_3^3 + 6\alpha_4 x_1 x_2 x_3 = 0.$$

Every curve goes into itself by the transformation. Any two curves of the system intersect in nine points, making three triads. Now choose any two points $(x)$, $(\bar{x})$ as basis points of the system of cubic curves. Associated with $(x)$ are two distinct points forming a triad, and associated with $(\bar{x})$ are two points so that the six points form two triads of permuted points. There remains one triad of variable points which are rationally distinct and are images of a point in $(y)$.

*Geometric Interpretation of Cyclic Type III.*

21. The relation between any point of the $(y)$ plane with its three corresponding images in the $(x)$ plane has been developed for Type III by Bottari.*

*Cyclic Type IV.*

22. Type IV is defined by a system of curves of order 9 which are invariant under the transformation and on which lie the three variable points which are permuted among themselves.

The system has $\infty^6$ degrees of freedom as shown in the form

$$\lambda_1 \psi_1 + C_6 (\lambda_2 \phi_1 + \lambda_3 \phi_2) + \lambda_4 \phi_1^3 + \lambda_5 \phi_1^2 \phi_2 + \lambda_6 \phi_1 \phi_2^2 + \lambda_7 \phi_2^3 = 0. \qquad (1)$$

Since $\psi$, $\phi_i \phi_k$ are invariant under $T$, but $C_6$ is not, we may reject the terms involving $C_6$,

$$\lambda_1 \psi + \lambda_2 \phi_1^3 + \lambda_3 \phi_1^2 \phi_2 + \lambda_4 \phi_1 \phi_2^2 + \lambda_8 \phi_2^3 = 0.$$

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* *Loc. cit.*, p. 282.

Two of the four constants in equation (2) may be chosen arbitrarily. We shall let one constant determine a point $P$ on the $\phi_2$ which also uniquely determines the two remaining associated points of the triad. Substituting the coordinates of this point in (2) there will be a fixed value $\lambda_1\psi_p$ and $\lambda_4\phi_p^3$ since $P$ does not lie on $\psi$ or $\phi_1$. The resulting equation is then, $\lambda_1 k+\lambda_4 l=0$ or $\lambda_4=-\lambda_1\frac{k}{l}$. A second constant may be used to determine a point $Q$ on $\phi_1$ with which are also associated the two remaining points of the triad.

$$\lambda_1\psi+\phi_1^2\phi_2+\phi_1\phi_2^2-\lambda_1\frac{k_1}{l}\phi_1^3-\lambda_1\frac{k'}{l'}\phi_2^3=0,$$

or

$$\lambda_1(\psi+\phi_1^3+\phi_2^3)+\lambda_2\phi_1^2\phi_2+\lambda_3\phi_1\phi_2^2=0.$$

The curves $f_i(x)=0$ intersect into two variable triads.

The curves $f_i(x)$ intersect in six fixed points which are fundamental points. $9\cdot 9-8\cdot 3\cdot 3-3=6$ composed of two triads of points $P_i$ and $Q_i$.

Through the point $(1, 0, 0)$ we have a pencil of lines of the form $\phi_1\phi_2(\phi_1 y_2+\phi_2 y_3)=0$. To the lines of the pencil correspond cubic curves in $(x)$. The transformation curves $\rho y_i=f_i(x)$ with eight triple points and two triads of simple points have fundamental elements in $(y)$ corresponding to their basis points. To $P_i$ correspond lines $p_i$ in $(y)$ through $(1, 0, 0)$. The residual image of these lines is a curve of order 9 composed of a sextic, the image of the point, and a cubic, the image of the line. In the same way the residual image of $Q_i$ is a cubic curve $\overline{Q}_i(x)$.

The triple points $R_i$ on the $f_i(x)$ have for images rational curves of order 3 in $(y)$ with a double point at $(1, 0, 0)$. The complete image in $(x)$ is a composite curve $r_i$ of order 27 containing the image of the fundamental point $(1, 0, 0)$ twice. The residual component $\bar{R}_i$ is of order 15, having eight points of multiplicity 5 at $R_i$.

The Jacobian of the system is a sextic having eight double points. It is of genus 2.

The curve $L$ is also a sextic of genus 2. There are four branches of the curve through $(1, 0, 0)$ consisting of two sets of tangents which count for eight intersections. The common tangent for two branches can not cut the curve again, but an arbitrary line through $(1, 0, 0)$ cuts $L$ in two other points corresponding to the two points of intersection of a cubic of the pencil with $K$. Two cubics do not meet $K$ except at fundamental points.

$L$ cuts $p_i$ in two points not at $(1, 0, 0)$ and $\overline{P}_i$ has two intersections with $K$ not at fundamental points. $6\cdot 3-8\cdot 1\cdot 2=2$. If $p_i$ is tangent to $L$ at

$(1, 0, 0)$, $\overline{P}_i$ has no simple intersection with $K$ but one tangency. This is also true for $q_i$ in $(y)$ and $\overline{Q}_i$ in $(x)$. $L$ cuts $r_i$ in ten points or five contacts. $6 \cdot 3 - 4 \cdot 2 = 10$. $K$ is tangent to each branch of $R$ at $R_i$ which correspond to the tangency of $L$ with $r_i$ at the five points, and there are no simple intersections. $6 \cdot 15 - 8 \cdot 5 \cdot 2 - 2 \cdot 5 = 0$.

*The Cyclic Transformation $N_3$.*

23. The cyclic transformation of period 3 involving eight fundamental points is of order 13. By means of the transformation the image of a line is a curve of order 13 on which the eight fixed points are distributed as follows: one triad of four-fold points $A, B, C$; one triad of five-fold points $\overline{A}, \overline{B}, \overline{C}$; one triple point $G$, and one six-fold point $\overline{G}$. In order to obtain the relation of the eight $P_i$ under the transformation we shall consider the two planes $(x)$ and $(x')$ on which are the points $A_4B_4C_4\overline{A}_5\overline{B}_5\overline{C}_5G_3\overline{G}_6 : A'_4B'_4C'_4\overline{A}'_5\overline{B}'_5\overline{C}'_5G'_3\overline{G}'_6$, respectively.

The image of $A$, $B$, and $C$ are three rational quartic curves in $(x')$ having three double points and five simple points, the image of $\overline{A}$, $\overline{B}$, and $\overline{C}$ are three rational curves of order 5 having six double points and two simple points; the image of $G$ is a cubic curve with one double point and six simple points, and of $\overline{G}$ the image is a sextic curve having one triple point and seven double points.

The multiplicity of $A$, $B$, $C$ on the Jacobian is 11, of $\overline{A}$, $\overline{B}$, $\overline{C}$ is 14, of $G$ is 8 and of $\overline{G}$ is 17.

By the transformation the line $A'C'$ goes into the curve

$$C_{13}(A_4B_4C_4\overline{A}_5\overline{B}_5\overline{C}_5G_3\overline{G}_6).$$

The image of a point $A'$ is the curve $C_4(A_1B_1C_1\overline{A}_1\overline{B}_2\overline{C}_2G_1\overline{G}_2)$ and of the point $C'$ is the curve $C_4(A_1B_1C_1\overline{A}_2\overline{B}_2\overline{C}_1G_1\overline{G}_2)$. Hence the residual image of the line $A'C'$ is a quintic curve $C_5(A_2B_2C_2\overline{A}_2\overline{B}_1\overline{C}_2G_1\overline{G}_2)$.

In the same way the residual image of $A'B'$ is the curve

$$C_5(A_2B_2C_2\overline{A}_2\overline{B}_2\overline{C}_1G_1\overline{G}_2).$$

Consider the points $(x')$ superposed upon the $(x)$ points such that $A'B'C'\overline{A}'\overline{B}'\overline{C}'G'\overline{G}' \equiv \overline{B}\overline{C}\overline{A}BCA\overline{G}G$, respectively.

By $T$ the point $A'$ goes into

$$C_4(A_1B_1C_1\overline{A}_1\overline{B}_2\overline{C}_2G_1\overline{G}_2) \equiv C_4(\overline{C}'_1\overline{A}'_1\overline{B}'_1C'_1A'_2B'_2\overline{G}'_1G'_2).$$

By $T^2$ this curve becomes a composite curve of order $4 \cdot 13$ or

$$C_{52}(A_{16}B_{16}C_{16}\overline{A}_{20}\overline{B}_{20}\overline{C}_{20}G_{12}\overline{G}_{24}),$$

from which the components which are images of the points $\overline{C}'\overline{A}'\overline{B}'C'A'B'\overline{G}'G'$ are deducted as $C_{47}(A_{14}B_{14}C_{14}\overline{A}_{18}\overline{B}_{18}\overline{C}_{18}G_{10}\overline{G}_{23})$.

Hence the residual component is a curve

$$C_5(A_2B_2C_2\overline{A}_2\overline{B}_2\overline{C}_1G_2\overline{G}_1)\equiv C_5(\overline{C}_2'\overline{A}_2'\overline{B}_2'C_2'A_2'B_1'\overline{G}_2'G_1').$$

By $T$ again the quintic becomes a composite curve $C_{65}(A_{20}B_{20}C_{20}\overline{A}_{25}\overline{B}_{25}\overline{C}_{25}G_{15}\overline{G}_{30})$, from which the image of the fundamental points of the quintic are deducted as before.

Hence this curve passes through the point $A'=\overline{B}$ once more than the parameters provide for, so that this point which is the residual image of $T^3$ is the original point. The transformation is, then, of period 3.

In the same way it can be shown that the point $\overline{G}'\equiv G$ by $T$ goes into $C_8(A_1B_1C_1\overline{A}_1\overline{B}_1\overline{C}_1G_0\overline{G}_2)$. By $T^2$ the residual image is $C_6(A_2B_2C_2\overline{A}_2\overline{B}_2\overline{C}_2G_2\overline{G}_2)$, and by $T^3$ the image is composed of the curve $C_{78}(A_{24}B_{24}C_{24}\overline{A}_{30}\overline{B}_{30}\overline{C}_{30}G_{18}\overline{G}_{36})$ and the point $G$, so that the image of $\overline{G}'\equiv G$ is the same point.

Also the point $\overline{A}'=B$ goes into a curve $C_5(A_1B_2C_2\overline{A}_2\overline{B}_2\overline{C}_2G_1\overline{G}_2)$ by $T$, and by $T^2$ the residual image is $C_4(A_2B_2C_1\overline{A}_1\overline{B}_1\overline{C}_1G_2\overline{G}_1)$. Under $T^3$ the image is the curve $C_{52}(A_{16}B_{16}C_{16}\overline{A}_{20}\overline{B}_{20}\overline{C}_{20}G_{12}\overline{G}_{24})$ and the point $\overline{A}'=B$, so that again $T^3$ is an identity.

CORNELL UNIVERSITY, *June*, 1917.

# *On Surfaces Containing a System of Cubics that do not Constitute a Pencil.*

By C. H. Sisam.

1. It is the object of this paper to classify completely the types of algebraic surfaces which are generated by an algebraic system $\gamma$ of $\infty^1$ cubic curves which do not constitute a pencil (so that two generic curves of the given system intersect in $\nu \geq 1$ variable points), and to point out a few salient properties of surfaces of the class that belong to types not already known.

## I. The Given Surface is Rational.

2. Since a rational surface possesses no integrals of total differentials of the first kind, the cubics of the given system are contained in a linear system of order $\nu$ and dimension $r \geq 2$ of cubic curves.*

3. If the cubics of the linear system to which the given systesm $\gamma$ belongs are rational, we can represent the given rational surface on a plane in such a way that, to the linear system of cubics corresponds a linear system of curves, of some order $m$, which have a fundamental $(m-1)$-fold point and $t \leq m-1$ simple fundamental points.† Since the given linear system is of order $\nu$, we have

$$m^2-(m-1)^2-t=\nu,$$

and since $t \leq m-1$, we have $m \leq \nu$.

4. If $\nu=1$, we have $m=1$, so that the curves corresponding to the cubics are the right lines in the plane. To the plane or hyperplane sections of the surface correspond curves which intersect the lines of the plane in three points. Hence, *if* $\nu=1$, *the surface generated by the given system of cubics is the surface of order* 9, *belonging to space of nine dimensions, which is represented parametrically by the cubics in a plane, or it is the projection of such a surface.*

* Humbert, "Sur quelques points de la théorie,".... *Journal de Mathématiques*, Ser. 4, Vol. X (1894), p. 195.

† Guccia, *Rendiconti di Palermo*, Vol. I (1887), p. 139. One notices here that a system of curves of odd order on the given surface can not be represented by a system of conics not having a fundamental point, since the curves representing the plane or hyperplane sections must intersect these curves in an odd number of non-fundamental points.

Conversely, on a surface of this type, the cubics corresponding to the tangents to an algebraic curve in the parametric plane constitute a system $\gamma$ of the required type. This surface was discussed by P. del Pezzo, in the *Rendiconti di Palermo*, Vol. I (1887), p. 241.

5. *If $\nu>1$, and if the cubics of the linear system to which the system $\gamma$ belongs are rational, the surface is a ruled surface of order not greater than* 4. In fact, if $\nu=2$, then $m=2$ so that the curves corresponding to the cubics of the linear system are conics through two fixed points $P_1$ and $P_2$. Let the curves corresponding to the plane sections be of order $n$ and of multiplicity $\sigma_1$ and $\sigma_2$ at $P_1$ and $P_2$, respectively. Then

$$2n-\sigma_1-\sigma_2=3.$$

Since $\sigma_1+\sigma_2\leq n$, we have $n\leq 3$. It is no restriction to suppose that $n=3$, since the case $n=2$ can be transformed birationally into this one. Then $n=3$, $\sigma_1=2$, $\sigma_2=1$. The surface defined parametrically by such a system of curves is a ruled quartic, belonging to a space of five dimensions, or it is the projection of such a surface.

If $\nu=3$, we find by similar reasoning that the surface is a ruled cubic, belonging to a space of four dimensions, or it is the projection of such a surface.

Finally, if $\nu\geq 4$, the surface is a quadric or a plane.*

6. If a generic cubic of the linear system to which $\gamma$ belongs is not rational, then the surface is either a cubic surface or a plane. For, if the order of the rational surface containing the system $\gamma$ exceeds 4, then, by the reference just cited, $\nu=1$ and the cubics of the linear system are rational since their points can be put in (1,1) correspondence with the curves of a linear pencil in the linear system. If the order of the surface equals 4, and the cubics of the linear system are plane cubics, the residual sections of their planes constitute a pencil of right lines on the surface. The cubics are thus rational, since their points are in (1, 1) correspondence with the generators of a rational ruled surface.

## II. THE SURFACE IS NON-RATIONAL AND OF ORDER NOT GREATER THAN 5.

7. Since the given surface is not rational, a generic cubic of $\gamma$ is of genus 1† and lies in a plane. If the given surface is of order 3, it is a non-singular cone and the system $\gamma$ is formed by the sections of the cone by an algebraic system of $\infty^1$ planes. If the surface is of order 4, the residual sections by the

* Cf. the author, "Nouvelles Annales de Mathématiques," Ser. 4, Vol. XVII, 1918.

† Castelnuovo-Enriques, "Sopra alcune questioni,".... *Annali di Matematica*, Ser. 3, Vol. VI (1901), p. 213.

planes of the cubics are right lines. The surface is thus a ruled quartic of genus 1, and the system $\gamma$ is cut from it by an algebraic system of $\infty^1$ tangent planes.

8. If the given surface is a quintic, the residual section by a plane or hyperplane containing a cubic of $\gamma$ is a conic. If a generic residual conic is not composite, the surface is generated by a non-rational pencil of conics. Such a surface* has three concurrent double lines and a tacnode. The cubics $\gamma$ lie in the planes which pass through the tacnode and are tangent to the surface. This surface is normal in three dimensions.

If a generic residual conic is composite, the quintic is a ruled surface of genus 1 belonging to a space of four dimensions or it is the projection of such a surface. The cubic curves on a given ruled quintic surface of genus 1 belonging to $S_4$ constitute the residual intersections of the surface with the $S_3$ defined by two generators. Since these cubics constitute a system $\infty^1$, and intersect in one variable point, they constitute a system $\gamma$.

9. *No ruled surface that contains a system $\gamma$ of cubics is of order greater than* 5. For the generators of such a surface set up, between the points of two generic cubics of $\gamma$, a $(1,1)$ correspondence in which at least one point, common to the two cubics, is self-corresponding since it is not multiple on the surface. The order of the surface defined by such a correspondence does not exceed 5.

10. In the remaining cases, we shall transform the given surface birationally into a ruled quintic surface $\phi$ of genus one, belonging to $S_4$. We point out here, for use in this connection, some properties of such surfaces $\phi$.

11. *Precisely* $\frac{5x(x+1)}{2}$ *independent linear conditions must be satisfied by the coefficients in the equation of an hypersurface $H_1^x$ of order $x$, in $S_4$, in order that it contain a given surface $\phi$.*

This theorem is true for $x=1$, since $\phi$ does not lie in an $S_3$. We assume it true for all orders less than the given order $x$. Since the rectilinear generators of $\phi$ intersect each cubic of $\gamma$, a necessary and sufficient condition that $H^x$ contains $\phi$ is that it contains $x+1$ generic cubics $C_1, C_2, \ldots, C_{x+1}$ of $\gamma$. An $H^x$ contains the elliptic cubic $C_1$ if it contains $3x$ generic points on $C_1$. It then contains $C_2$ if it contains $3x-1$ generic points on $C_2$, etc.

These $3x+3x-1+\ldots.+2x=\frac{5x(x+1)}{2}$ linear conditions on the coefficients in the equation of an $H^x$ are independent. For, there exists an $H^x$ which

*Cf. the author, American Journal of Mathematics, Vol. XXX (1908), pp. 115–116.

contains $C_1, C_2, \ldots, C_{j-1}$ (where $j$ has any of the values $1, 2, \ldots, x+1$) and which also contains $3x-j$ generic points $P_j, P_{j+1}, \ldots, P_{3x-1}$ on $C_j$ but which does not contain $C_j$. There exists, in fact, an hypersurface $H_1^{x-1}$, of order $x-1$, which contains $C_1, C_2, \ldots, C_{j-2}$ and $P_j, P_{j+1}, \ldots, P_{3x-4}, P_{3x-3}$ and an hyperplane $H_1'$ that contains $C_{j-1}$ and $P_{3x-2}$. Similarly, there exists a second hypersurface $H_2^{x-1}$, that contains $C_1, C_2, \ldots, C_{j-2}$ and $P_j, P_{j+1}, \ldots, P_{3x-4}$, $P_{3x-2}$ and a second hyperplane $H_2'$ that contains $C_{j-1}$ and $P_{3x-3}$. Since the points $P$ are generic, the composite hypersurface $H_1^{x-1}H_1'$ intersects $C_j$ in two points that do not lie on $H_2^{x-1}H_2'$. Thus, no hypersurface of the pencil $\lambda_1 H_1^{x-1}H_1' + \lambda_2 H_2^{x-1}H_2' = 0$ contains $C_j$. All the hypersurfaces of this pencil contain $C_1, C_2, \ldots, C_{j-1}$ and $P_j, P_{j+1}, \ldots, P_{3x-2}$. One hypersurface of the pencil also contains $P_{3x-1}$.

12. Any curve on $\phi$ that intersects a generic generator in $x$ points and a generic cubic of $\gamma$ in $y$ points is of order $y+2x$, since it intersects the $S_3$ containing two generic generators and a cubic of $\gamma$ in $y+2x$ points.

13. *For no curve on $\phi$ can $x$ be greater than $2y$.* For, the order of such a curve would be $n = y+2x < \frac{5x}{2}$. Since $\frac{5x(x-1)}{2} - n(x-1) > 0$, such a curve would lie on an $H^{x-1}$ which does not contain $\phi$ but has $x$ points in common with each generator.

14. Any curve on $\phi$ for which $2y = x$ lies on an $H^x$, since

$$\frac{5x(x+1)}{2} - \frac{5x}{2} \cdot x > 0.$$

The residual intersection consists of $5x - \frac{5x}{2} = \frac{5x}{2}$ generators of $\phi$, since it does not intersect a generic generator.

There are three curves on $\phi$ for which $x=2, y=1$. For, put the cubics of $\gamma$ into $(1,1)$ correspondence with the points of a plane cubic curve $C$. Then each point of $\phi$ corresponds to the pair of points on $C$ defined by the two cubics of $\gamma$ through $P$. Then the points of a rectilinear generator on $\phi$ correspond to the pairs of a linear series $g_2^1$ on $C$ and the points of a cubic of $\gamma$ to the pairs for which one point is fixed. There are three irrational involutions of order 2 on $C$ defined by the three cubics of which $C$ is the Hessian-Steinerian. Two pairs of such an involution belong to a given $g_2^1$ and one pair contains a given point $P$. The points on $\phi$ defined by these three involutions thus constitute three quintic curves, $D_1, D_2, D_3$ of genus 1, for which $x=2, y=1$.

We shall show (Art. 35) that there is a rational pencil of curves on $\phi$ for which $x=4, y=2$. Each of the curves $D_1, D_2, D_3$ counted twice is a curve of the

pencil. Every curve on $\phi$ for which $x=2y$ degenerates into curves of this pencil together, possibly, with one or more of the curves $D$. For let $C$ be such a curve. If $C$ is composite, each component intersects the generators in twice as many points as it does the cubics of $\gamma$ so that $C$ lies on a proper or composite $H^z$ and forms, with $\frac{5x}{2}$ generators, the complete intersection of $H^z$ with $\phi$. Let $P$ be a generic point on $C$ and let $C'$ be the curve of order 10 of the above pencil through $P$. Then $C'$ has in common with $H^z$ the point $P$ and $10x$ points on the generators common to $\phi$ and $H^z$. Hence $C'$ (or a component of it, if $P$ lies on a quintic $D$) lies on $H^z$. It forms a component of $C$ since, otherwise, $H^z$ would have more than $x$ points in common with a generic generator of $\phi$.

### III. The Surface is Non-rational and of Order not Less than 6.

15. Since the order of the surface exceeds 5, the surface is not ruled (Art. 9). The system $\gamma$ is of order unity (Art. 5, footnote) and index 2.* It is thus representable by pairs of points on a plane cubic curve (cf. Art. 14) and is of genus $p_g=0$, $p_n=-1$.

16. Let the given surface $F$ be projected, if necessary, into a surface $F'$, of order $m$, in $S_3$. Then the cubics on $F'$ that constitute the projections of the cubics of $\gamma$ on $F$ intersect the adjoints to $F'$ of order $m-3$ in just two points which are not multiple on $F'$. Suppose, in fact, that the cubics do not have a fixed point in common. Since consecutive cubics intersect, the plane of a generic cubic $C$ touches $F'$ at only two of the intersections of $C$ with the plane of the consecutive cubic. The remaining $3m-11$ points common to $C$ and the residual section of its plane with $F'$ lie on the multiple curve and thus on the adjoints of order $m-3$. Similarly, if the cubics have a fixed point $P$ in common, the plane of $C$ touches $F'$ at one point on $C$. The remaining intersections of $C$ with the residual curve, except one at $P$, are common to the adjoints of order $m-3$.

17. *The adjoints of order $m-3$ constitute a linear system of at most two dimensions.* Suppose, in fact, that they constituted a linear system of $r>2$ dimensions. Since the cubics are not rational, they determine a linear series $g_2^1$ on each cubic of $\gamma$. Then the $\infty^{r-3}$ adjoints of order $m-3$ through two generic points of one cubic and one generic point of a second generic cubic would contain all the cubics and would thus contain $F'$. But this is impossible, since the order of the adjoints is less than that of $F'$.

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* Castelnuovo-Enriques, *Mathematische Annalen*, Vol. XLVIII, p. 314.

18. *The genus of the plane sections of $F'$ does not exceed* 4. Let $\Pi$ be the genus of a generic plane section of $F'$ and let $r$ be the dimension of the linear system of adjoints of order $m-3$. Then*

$$r=\Pi-1-(p_g-p_n) \quad \text{or} \quad \Pi=r+1+p_g-p_n.$$

But $r\leq 2$, $p_g=0$, $p_n=-1$, so that $\Pi\leq 4$. Moreover, $\Pi\geq 3$, since the surface is not ruled.†

19. If $\Pi=3$, it is known‡ that the surface contains an irrational pencil of conics. The quartics cut from such a surface $F'$ by the pencil of adjoints of order $m-3$ degenerate into pairs of conics.

If $\Pi=4$, then $r=2$ so that a pencil of adjoints of order $m-3$ pass through a generic point $P$ on $F'$. All the adjoints of this pencil pass through two points $P_1$ and $P_2$ fixed by $P$ on the cubics $C_1$ and $C_2$, respectively, through $P$. In case $P_1$ and $P_2$ lie on a cubic $C$ of $\gamma$, they are corresponding points in the $g_2^1$ defined on $C$ by the adjoints. Then the adjoint surface that contains $C$ contains all the pairs of the $g_2^1$ on $C$ and thus contains a rational cubic curve (locus of $P$) which forms, with $C$, the variable sextic of intersection of the adjoint with $F'$. In case $P_1$ and $P_2$ do not lie on the same cubic, let $C_1'$ and $C_2'$ be cubics of $\gamma$ through $P_1$ and $P_2$, respectively. The pencil of adjoints through $P_1$ intersect $C_1'$ in $P_1$ and in a second fixed point distinct from its intersection with $C_2$. It follows that the adjoint surface that contains $C_2$ has $C_1'$ for its residual intersection and, similarly, that the one containing $C_1$ has $C_2'$ for its residual intersection.

20. Let the system $\gamma$ of cubics on a surface $F$, belonging to three or more dimensions, be put on $(1,1)$ correspondence with the system of cubic curves on a ruled quintic surface $\phi$, of genus 1, belonging to $S_4$. Then a $(1,1)$ correspondence is set up between $F$ and $\phi$ by taking two points as corresponding when they lie at the intersection of corresponding pairs of cubics.

21. Let $x>1$ be the order of the rational curves on $F$. Then the plane or hyperplane sections of $F$ transform into a linear system of curves on $\phi$ that are of order $3+2x$ since they intersect each cubic in three points and each generator in $x$ points and thus intersect any $S_3$ which contains a cubic and two generators in $3+2x$ points. Every such curve lies on an $H^x$, since $\frac{5x(x+1)}{2}$ $-x(2x+3)>0$ when $x>1$. The residual intersection of the $H^x$ with $\phi$ consists of $3(x-1)$ generators. The plane or hyperplane sections of $F$ thus correspond to sections of $\phi$ by a linear system of $H^x$ through $3(x-1)$ fixed generators.

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*Picard et Simart, "Théorie des Fonctions Algébriques," Vol. II, p. 489.

†Castelnuovo-Enriques, *Mathematische Annalen*, Vol. XLVIII (1897), p. 308.

‡Scorza, *Annali di Matematica*, Ser. 3, Vol. XVI, p. 255, *et seq.*

*The order of the rational curves on F does not exceed* 4. For, since $y=3$, we have at once $x \leq 5$ (Arts. 13, 14). But the $H^5$ through twelve generators define on $\phi$ a linear system of only two dimensions so that $x \leq 4$.

a. *The Rational Curves on F are Conics.*

22. Since $x=2$ and $y=3$, the curves on $\phi$ corresponding to the plane or hyperplane sections of $F$ are of order 7 and constitute the residual intersection of $\phi$ with a system of $H^2$ through three fixed lines. The complete linear system to which they belong is of order 8 and dimension 5. Hence the surface $F$ is of order 8 and belongs to a space of five dimensions, or it is the projection of such a surface. This surface is the "first type" discussed by Scorza in an article entitled "Le superficie a curve sezioni di genere 3" in the *Annali di Matematica*, Ser. 3, Vol. XVII (1910), p. 320.

b. *The Rational Curves on F are Cubics.*

23. To the system of plane or hyperplane sections of $F$ corresponds a linear system of curves of order 9 cut from $\phi$ by $H^3$ through six fixed lines. The complete linear system defined by these curves is of order 9 and dimension 5. Hence, *the given surface F is of order* 9 *and belongs to a space of five dimensions, or it is the projection of such a surface.* A generic hyperplane section is of genus 4 (Arts. 18 and 19).

24. Let $F$ belong to $S_5$. Since two generic cubics of $\gamma$ intersect, they lie in an $S_4$. The residual intersection of this $S_4$ with $F$ is a cubic of $\gamma$ since it corresponds to a cubic on $\phi$. Since any curve on $F$ intersects three such cubics of $\gamma$ in the same number of points, we deduce that: *the order of any curve on F is equal to* $3y$, where $y$ is the number of its intersections with a generic curve of $\gamma$.

25. There are three types of cubic curves on $F$, defined by the number $x$, of their intersections with the rational curves on $F$. If $x=0$, the curves are the rational cubics; if $x=1$, they are the curves of $\gamma$; if $x=2$, there are just three curves $D_1'$, $D_2'$, $D_3'$. They are of genus 1, and correspond to the three quintics on $\phi$ (Art. 14) that intersect the generators in two points. The planes of two such cubics $D'$ do not intersect. Otherwise, they would lie in an $S_4$ which would have four points in common with each rational cubic on $F$.

26. The rational cubics define a non-rational involution of order 2 on each cubic $D'$. The lines joining corresponding points of such an involution envelope a curve of class 3 in the plane of $D'$. It follows that the $S_3$ defined by the rational cubics on $F$ generate an hypersurface of order 6, since an $S_3$ which contains a generic line $l_1$ in the plane of $D_1'$ and $l_2$, in the plane of $D_2'$, intersects

the locus of the $S_3$ defined by the rational cubics in a ruled surface which has $l_1$ and $l_2$ as triple directrices and is thus of order 6. The point in which the $S_3$ intersects the plane of $D_3'$ is a triple point on the sextic. Hence the generator through that point is a triple generator on the ruled surface. It follows that the cubic threespread formed by the lines that intersect the planes of $D_1'$, $D_2'$ and $D_3'$ is a triple threespread on the sextic hypersurface. The hypersurface is of genus 1, and has no other multiple points.

27. Let $S$ be the threespace of a generic rational cubic $C$ on $F$. The plane of a cubic $D'$ has a line in common with $S$ and determines, with $S$, an $S_4$ whose residual intersection with $F$ is a cubic of $\gamma$ whose plane has a line in common with $S$. Conversely, if $\Gamma$ is a cubic of $\gamma$ whose plane has a line in common with $S$, then $\Gamma$, $C$ and a cubic $D'$ lie in an $S_4$. Each of the two intersections of $\Gamma$ with $S$ that do not lie on $C$ lies on a cubic $D'$. Moreover, the threespread generated by the planes of the cubics $\gamma$ is of order 9, since it intersects $S$ in three right lines and in the cubic $C$ which is counted twice since two cubics $\gamma$ pass through each point of $C$.

28. The $S_4$ that intersect $F$ in three cubics of $\gamma$ (Art. 24) define a $g_3^2$ on the system $\gamma$. Let these $S_4$ be put in (1, 1) correspondence with the points of a plane in such a way that the $S_4$ corresponding to the points on a line define a $g_3^1$ on $\gamma$. Each cubic of $\gamma$ belongs to one group of such a $g_3^1$, so that each point of $F$, and thus each point of $S_5$, lies in two of the $S_4$ define by the points of a line. Then the $\infty^1 S_4$ that pass through a given point correspond to the points of a conic or: the locus of the $S_4$ that intersect $F$ in three cubics of $\gamma$ is the dual of a surface of Veronese.

29. The sextics on $F$ intersect the cubics $\gamma$ in two points (Art. 24). Those that intersect the rational cubics once, correspond to the quartic curves on $\phi$. They constitute $\infty^1$ bundles, intersect in three points and are of genus 1. Three of these bundles constitute the residual intersections of the bundles of $S_4$ containing a cubic $D'$. A fourth bundle constitutes the adjoint sextics (Art. 19) to the hyperplane sections of $F$.

The sextics that intersect the rational cubics twice are of genus 2, intersect in four points and constitute the $\infty^1$ bundles forming the residual intersections of the $S_4$ that contain a cubic $\gamma$. Those that intersect the rational cubics in three points are of genus 2, intersect in three points and constitute the $\infty^1$ pencils residual to the rational cubics. Those that intersect the rational cubics in four points are of genus 1 and constitute a pencil (Art. 14).

30. The projection of $F$ on an $S_3$ from a generic line $l$ in the plane of a cubic $C$ of $\gamma$ is a sextic surface with a tacnode at the intersection of the plane

of $C$ with $S_3$. The projections of the cubics of $\gamma$ pass through the tacnode and lie in pairs in the planes tangent to a quadric cone. The rational cubics also pass through the tacnode. Three of them are nodal, and coplanar with the projections of the cubics $D'$. The surface has three coplanar double lines, projections of cubics of $\gamma$ that intersect $l$. The points of intersection of these double lines are triple points on the surface. The residual nodal cubic passes through these triple points.

31. The projection of $F$ on an $S_3$ from a generic line $l$ in the plane of a cubic $D'$ is a sextic surface with a triple point at the intersection of the plane of $D'$ with $S_3$. The rational cubics have a node at this triple point. Their planes envelope a cone of class 3. These planes also contain the cubics $\gamma$. The six cubics of $\gamma$ that intersect $l$ project into double lines forming the six edges of a tetrahedron.

### c. *The Rational Curves on F are Quartics.*

32. To the system of plane or hyperplane sections of $F$ corresponds a linear system of curves of order 11 cut from $\phi$ by $H^4$ through nine fixed lines. The complete linear system defined on $\phi$ by these curves is of order 8 and dimension 4. Hence, *the given surface F is of order* 8 *and belongs to four dimensions, or it is the projection of such a surface.* A generic hyperplane section is of genus 4 (Arts. 18 and 19).

33. Let $F$ belong to an $S_4$. An $H^4$ that defines a curve on $\phi$ corresponding to an hyperplane section of $F$, has eighteen of its twenty intersections with a quintic $D$ (Art. 14) fixed on the nine fundamental lines. These $H^4$ thus determine a $g_2^1$ on each quintic $D$ so that *the surface F has three double lines* $d_1, d_2, d_3$. No two of these double lines can intersect. Otherwise, each $S_3$ of the pencil containing them would have for residual intersection with $F$ a rational quartic (since it corresponds to a right line on $\phi$) from which it would follow that $F$ is rational.

34. The rectilinear generators of $\phi$ determine on each quintic $D$ a non-rational involution of order 2 which has two pairs of points in common with the $g_2^1$ defined by the curves corresponding to the hyperplane sections of $F$. It follows that two rational quartics on $F$ have a double point on each double line.

Let $S$ be the threespace defined by $d_2$ and $d_3$. Its residual intersection with $F$ is a rational quartic $C_1$. The two intersections of $C_1$ with $d_1$ coincide at the intersection of $d_1$ with $S$. Hence $C_1$ has a double point on $d_1$. Similarly, $d_1$ and $d_3$ define a quartic $C_2$ with a double point on $d_2$ and $d_1$, $d_2$ a quartic $C_3$ with a double point on $d_3$. The three double points are collinear.

Let $C$ be the rational quartic, other than $C_1$, that has a double point on $d_1$. Then, since $C$ does not lie in the $S_4$ containing $d_2$ and $d_3$, $C$ has a double point on a second double line $d_2$ and lies in a plane. This plane can not contain $d_3$. Otherwise, the $S_3$ through it would define a rational pencil of conics on $F$. Hence the third double point of $C$ lies on $d_3$.

35. The residual intersections with $F$ of the $S_3$ through the plane of $C$ constitute a rational pencil of quartics of genus 1 which intersect the rational quartics in four points and the cubics of $\gamma$ in two points. No two quartics of this pencil intersect. The curves on $\phi$ corresponding to these quartics are of order 10. They intersect the generators in four points and the cubics of $\gamma$ twice.

36. No rational quartic on $F$, other than $C$, $C_1$, $C_2$, $C_3$, lies in an $S_3$. Otherwise, let $C'$ be such a quartic. It has a double point on at least two double lines (since it can not lie in the $S_3$ defined by two double lines) and lies in a plane. The residual intersections of the $S_3$ containing this plane constitute a pencil of quartics, distinct from that of Art. 35, which intersect the rational curves four times and the cubics of $\gamma$ twice. This is impossible (Art. 14).

37. Let $\Gamma$ be any cubic of $\gamma$. The plane of $\Gamma$ intersects a given double line $d_1$ and defines with it an $S_3$ whose residual intersection with $F$ is a second cubic $\Gamma'$ of $\gamma$. The points in which $d_2$ and $d_3$ intersect $S_3$ lie on $\Gamma$ and $\Gamma'$ and thus on the line of intersection of their planes. It follows that each cubic of $\gamma$ is intersected in two points (which lie on two of the double lines) by each of three other cubics of $\gamma$.

Let $S$ be the threespace defined by $d_1$ and $d_2$. Four cubics of $\gamma$ pass through a generic point of $d_1$. The lines in which the planes of these cubics intersect $S$ coincide in pairs and intersect $d_2$. These lines thus establish a $(2, 2)$ correspondence between the points of $d_1$ and $d_2$ and generate a ruled quartic surface. The generators of this surface are bisecants of the rational curve $C_3$ (Art. 34). The common secant line of $d_1$, $d_2$, $d_3$ is a double generator of the ruled quartic.

The hypersurface generated by the planes of $\gamma$ is of order 8, since its section by the $S_3$ of two double lines is a ruled quartic counted twice. This octavic hypersurface has $F$ and the three ruled quartic surfaces defined by pairs of the lines $d_1$, $d_2$, $d_3$ as double surfaces. Since it is of genus 1, it has no other multiple points.

38. *The surface $F$ forms, with the plane of the rational plane quartic on it, the basis surface of a pencil of cubic hypersurfaces.* For, the $H^3$ that contain the three double lines and six generic cubics of $\gamma$ contain all the cubics of $\gamma$ and

thus contain $F$. That an $H^3$ contain four points on each double line is twelve conditions, that it further contain six generic cubics of $\gamma$ is twenty-one more conditions on the thirty-five homogeneous coefficients in the equation of the hypersurface. There thus exists a pencil of such $H^3$. The plane of the rational quartic $C$ (Art. 34) clearly lies on all these $H^3$ since every line in it has four points in common with each $H^3$.

39. Let a curve of order $n$ on $F$ intersect a generic rational quartic in $x$ points and a generic cubic of $\gamma$ in $y$ points. Then

$$4y=x+n.$$

For, the corresponding curve on $\phi$ is of order $y+2x$. Of its $4(y+2x)$ intersections with the $H^4$ that defines a generic hyperplane section of $F$, $9x$ lie on the nine fundamental lines (Art. 32). The remaining $4(y+2x)-9x=n$ define the intersections of the given curve with the given hyperplane. Since $y\leq 2x$, we have $x\leq n$. The equality sign holds (for non-composite curves) only for $x=2$ and $x=4$ (Art. 14).

If $n=2$, we have $x=2$, $y=1$. The curves are the three double lines.

If $n=3$, we have $x=1$, $y=1$. The curves are the cubics of $\gamma$.

If $n=4$ and $x=0$, $y=1$, the curves are the rational quartics.

If $n=4$ and $x=4$, $y=2$, the curves are the pencil of residual quartics in the $S_3$ through the plane of the rational quartic $C$ (Art. 34).

If $n=5$, we have $x=3$, $y=2$. There are $\infty^1$ pencils of these curves. They form the residual intersections of the $S_3$ that contain a cubic of $\gamma$. They are of genus 2 and intersect in three points.

If $n=6$, we have $x=2$, $y=2$. There are $\infty^1$ bundles of such curves. They are of genus 2 and intersect in four points. In each pencil in any bundle, there are two which break up into pairs of cubics of $\gamma$. One such bundle constitutes the bundle of adjoint sextics to the hyperplane sections of $F$ (Art. 19).

If $n=7$ and $x=1$, $y=2$, we have $\infty^1$ bundles of curves of genus 1 that intersect in three points. If $x=5$, $y=3$, we have $\infty^1$ bundles of curves of genus 3 that intersect in five points. All the curves of a bundle intersect each double line in a fixed point.

40. The projection of $F$ on an $S_3$ from a generic point on a double line $d$ is a sextic surface having a tacnode at the intersection of $d$ with $S_3$. The tacnodal tangent plane contains the projection of the rational plane quartic and two right lines. The cubics $\gamma$ lie in pairs in the tangent planes to a quadric cone with vertex at the tacnode. The rational quartics have a double point at the tacnode. The double curve is composed of six right lines forming the edges of a tetrahedron.

THE UNIVERSITY OF ILLINOIS, *June* 19, 1917.

# *An Isoperimetric Problem with Variable End-Points.*

By Archibald Shepard Merrill.

## *Introduction.*

The object of this paper is to give a complete discussion of the necessary and sufficient conditions for a maximum (minimum) for a type of problems in the Calculus of Variations which are closely related to the usual isoperimetric problems, and in which both end-points are allowed to vary along a given fixed curve. We suppose that we have given the fixed curve $L$ and a certain arc $E_{12}$

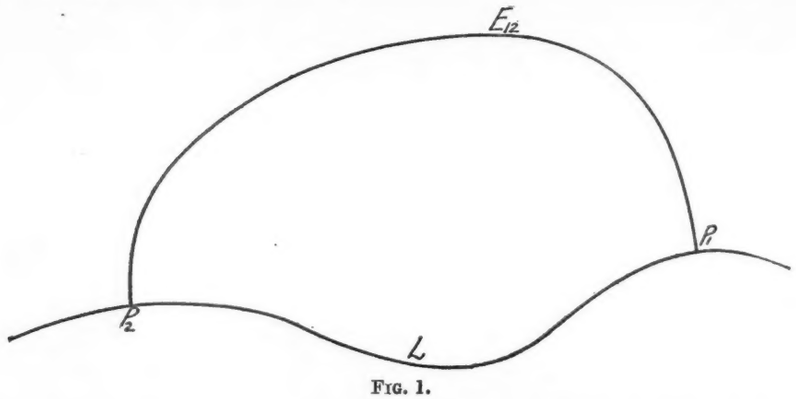


Fig. 1.

joining two points $P_1$ and $P_2$ of $L$. The problem is to determine the properties which the curve $E_{12}$ must have in order that the integral of a given function $F(x, y, x', y')$ along $L$ from $P_2$ to $P_1$, then along $E_{12}$ from $P_1$ to $P_2$ shall be a maximum (minimum), while the integral of a second function $G(x, y, x', y')$ along $E_{12}$ has a prescribed value. Thus the function $J$ to be maximized (minimized) is a sum of two integrals:

$$J = \int_{L_{21}} F(x, y, x', y')\, dx + \int_{E_{12}} F(x, y, x', y')\, dt,$$

while the integral

$$K = \int_{E_{12}} G(x, y, x', y')\, dt$$

is to remain fixed in value.

A familiar application of this type of problem is the well-known Problem of Dido. In this application the area included by the arc $P_1P_2$ of $E_{12}$ and the arc $P_2P_1$ of $L$ is to be a maximum, while the arc $P_1P_2$ of $E_{12}$ is to have a preassigned length.

For the problem at hand certain conditions of the corresponding isoperimetric problem with fixed end-points must hold, and these are already well known, viz., the Euler, Weierstrass, Legendre and Jacobi necessary conditions.* The transversality condition is also readily obtainable, and has been deduced for some special cases of the problem here discussed.† In the present paper a new necessary condition, corresponding to the Jacobi condition in other problems in the Calculus of Variations, will be deduced and discussed both for the case of one end-point variable and for the case of both end-points variable. In obtaining this we make use of the derivatives of the "extremal integral" for any isoperimetric problem, and Section 2 is given over entirely to the computation of these derivatives. Conditions which are sufficient for a maximum of $J$ when $K$ is fixed are readily obtained with the help of a theorem proved by Hahn. A geometric interpretation of the new condition is given in Section 5. Finally in Section 6, as an application of the general theory developed, a discussion of the above-mentioned Problem of Dido is given.

## § 1. *Conditions Deducible from Known Results.*

Consider a fixed curve

$$x=\tilde{x}(s), \quad y=\tilde{y}(s) \tag{L}$$

not intersecting itself, and two points $P_1(s=s_1)$ and $P_2(s=s_2)$ on $L$ with $s_2<s_1$. Let $E_{12}$ be an arc

$$x=\phi(t), \quad y=\psi(t), \quad t_1\leqq t\leqq t_2, \tag{E}$$

cutting $L$ at $P_1(t=t_1)$ and $P_2(t=t_2)$. The function to be maximized or minimized is then of the form

$$J=\int_{s_2}^{s_1}F(\tilde{x},\tilde{y},x',y')\,ds+\int_{t_1}^{t_2}F(\phi,\psi,\phi',\psi')\,dt,$$

while the integral

$$K=\int_{t_1}^{t_2}G(\phi,\psi,\phi',\psi')\,dt$$

is to remain constant in value.

For simplicity the following discussion will be restricted to the determination of a maximum for $J$. Consider the totality of arcs whose end-points

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* See Bolza, "Vorlesungen über Variationsrechnung, Chapter X.

† See Bolza, *loc. cit.*, p. 520.

lie upon $L$. In this class there is a sub-class $\mathfrak{M}$ of arcs which give the integral $K$ a fixed value $k$. The problem is then to find necessary and sufficient conditions that a particular arc $E_{12}$ of $\mathfrak{M}$, intersecting arc $L$ at $P_1(x=x_1)$ and $P_2(x=x_2)$, shall give to $J$ a larger value than any other arc of $\mathfrak{M}$ in a certain neighborhood of arc $E_{12}$.

It is presupposed that the arc $L$ is regular* in a neighborhood of the values $x_2 \leqq x \leqq x_1$. The class $\mathfrak{M}$ is further restricted to contain only regular arcs, and in particular the arc $E_{12}$ whose maximizing properties are to be investigated is assumed to be of class $C'''$.† It is also assumed that $E$ is not an extremal for the integral $K$.

The function $G$ is of class $C'''$ for all values $(x, y, x', y')$ for which $(x', y') \neq (0, 0)$ and $(x, y)$ is in a neighborhood of $E_{12}$, while $F$ is of the same class for $(x', y') \neq (0, 0)$ and $(x, y)$ in a neighborhood of those on $L_{12}+E_{12}$. Both these functions satisfy the usual homogeneity conditions

$$F(x, y, kx', ky') = kF(x, y, x', y'),$$
$$G(x, y, kx', ky') = kG(x, y, x', y')$$

in these neighborhoods for every $k>0$.

The necessary conditions that $J(E)$ be a maximum with respect to all curves of $\mathfrak{M}$ with the same end-points, and keeping $K$ fixed, must be fulfilled. Hence we have at once the usual Euler, Weierstrass, Legendre and Jacobi conditions referred to above. These may be stated as follows:

I. *Euler Condition.*—The curve $E$ must satisfy for a certain constant value $\lambda$ the Euler differential equations

$$H_x - \frac{d}{dt} H_{x'} = 0, \quad H_y - \frac{d}{dt} H_{y'} = 0, \tag{1}$$

where $H = F + \lambda G$. Such curves will, as usual, be called extremals.

II. *Weierstrass Condition.*—The Weierstrass $E$ function

$$E(x, y, p, q, x', y'; \lambda) = H(x, y, x', y'; \lambda) - x' H_{x'}(x, y, p, q; \lambda) - y' H_{y'}(x, y, p, q; \lambda)$$

must be greater than or equal to zero for all $(x, y, p, q; x', y')$, such that $(x, y, p, q)$ belongs to a point of $E_{12}$ while $(x', y')$ is different from $(0, 0)$.

III. *Legendre Condition.*—Along the arc $E_{12}$, $H_1 \leqq 0$, where

$$H_1 = \frac{H_{x'x'}}{y'^2} = -\frac{H_{x'y'}}{x'y'} = \frac{H_{y'y'}}{x'^2}.$$

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*An arc is said to be regular when it is continuous and consists of a finite number of arcs each of which has a well-defined and continuously turning tangent.

† See Bolza, *loc. cit.*, p. 13.

IV. Finally $E_{12}$ must satisfy the *Jacobi Condition* for fixed end-points; that is, the extremal arc $E_{12}$ must not contain in its interior either the point $P_1'$ conjugate to $P_1$, or the point $P_2'$ conjugate to $P_2$.

In the transversality condition there is a departure from the result obtained by Bolza * in a closely related problem. We proceed to its determination, however, in an analogous manner. Consider one end-point, say $P_1$, fixed. It is possible to set up in the usual way † a one-parameter family of variation curves

$$x=\phi(t,\alpha), \quad y=\psi(t,\alpha), \quad t_1\leqq t\leqq t_2,$$

which have the following properties. For $\alpha=\alpha_2$, $t_1\leqq t\leqq t_2$, the family contains the arc $E_{12}$. Furthermore, every arc passes through the point $P_1$ for $t=t_1$, and intersects the arc $L$ for $t=t_2$, which gives rise to the equations

$$\begin{aligned} \phi(t_1,\alpha)&=x_1, & \psi(t_1,\alpha)&=y_1,\\ \phi(t_2,\alpha)&=x(\alpha), & \psi(t_2,\alpha)&=\tilde{y}(\alpha). \end{aligned}$$

Finally, along each one of these curves, the integral $K$ has the assigned value $k$. Substituting this family of variation curves in the expression for $J$ we find

$$J(\alpha)=\int_{\alpha}^{\alpha_1} F(x,\tilde{y},x',\tilde{y}')\,dx+\int_{t_1}^{t_2} F(\phi,\psi,\phi',\psi')\,dt.$$

Following the procedure of Bolza we have the condition that at the point $P_2$,

$$F(\tilde{x},\tilde{y},\tilde{x}',y')-H_{x'}(\phi,\psi,\phi',\psi')\tilde{x}'-H_{y'}(\phi,\psi,\phi',\psi')\tilde{y}'=0,$$

and by a similar argument at $P_1$ we have the following result:

V. *Transversality Condition.*—The curve $L$ must cut $E_{12}$ transversally at $P_1$ and $P_2$, that is, at both these points the condition

$$F(\tilde{x},\tilde{y},x',\tilde{y}')-H_{x'}(\phi,\psi,\phi',\psi')\tilde{x}'-H_{y'}(\phi,\psi,\phi',\psi')\tilde{y}'=0 \qquad (2)$$

must hold.

## § 2. *Derivatives of the Extremal Integral.*

We suppose now that the arc $E_{12}$ satisfies the necessary conditions of the preceding section, and further that the Legendre and Jacobi conditions for fixed end-points hold in the so-called stronger form. This means that $H_1<0$ along the arc $E_{12}$, and that $P_1$ and $P_2$ are not conjugate points on $E_{12}$. As a result of the continuity properties of $F$ and $G$, and the fact that $H_1\neq 0$ along $E_{12}$, it is known ‡ that this arc may be imbedded in a family of extremals

$$x=\phi(t,\alpha,\beta,\lambda), \quad y=\psi(t,\alpha,\beta,\lambda) \qquad (3)$$

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* *Loc. cit.*, pp. 519, 520. † Bolza, *loc. cit.*, pp. 473, 474. ‡ Bolza, *loc. cit.*, pp. 468 ff.

which contains $E_{12}$ for values $\alpha_0, \beta_0, \lambda_0, t_1 \leqq t \leqq t_2$. The functions $\phi, \phi_t, \psi, \psi_t$ are of the class $C'$ in all their arguments in a neighborhood of these values. The constant $\lambda$ is the isoperimetric constant for each extremal.

Suppose now that $M_1$ and $M_2$ are any two points $(x_1, y_1)$ and $(x_2, y_2)$ sufficiently near to $P_1$ and $P_2$, respectively. The equations

$$\left.\begin{aligned} \phi(\tau_1, \alpha, \beta, \lambda) = x_1, \quad \phi(\tau_2, \alpha, \beta, \lambda) = x_2, \\ \psi(\tau_1, \alpha, \beta, \lambda) = y_1, \quad \psi(\tau_2, \alpha, \beta, \lambda) = y_2, \\ \int_{\tau_1}^{\tau_2} G(\phi, \psi, \phi', \psi')\, dt = k, \end{aligned}\right\} \tag{4}$$

have the initial solution

$$(\alpha, \beta, \lambda, \tau_1, \tau_2, x_1, y_1, x_2, y_2) = (\alpha_0, \beta_0, \lambda_0, t_1, t_2, x_{10}, y_{10}, x_{20}, y_{20}),$$

where $(x_{10}, y_{10})$, $(x_{20}, y_{20})$ are now the coordinates of $P_1, P_2$, respectively. Furthermore, since the Jacobi condition in its stronger form is satisfied by the arc $E_{12}$, it follows that the functional determinant of the left members for $\alpha, \beta, \lambda, \tau_1, \tau_2$ is different from zero at this initial solution. It reduces in fact, after suitable transformations, to the determinant $D(t_1, t_2)$* formed for $E_{12}$, which vanishes only when $P_1$ and $P_2$ are conjugate. Hence equations (4) have unique solutions of the form

$$\left.\begin{aligned} \alpha(x_1, y_1, x_2, y_2), \quad \beta(x_1, y_1, x_2, y_2), \quad \lambda(x_1, y_1, x_2, y_2), \\ \tau_1(x_1, y_1, x_2, y_2), \quad \tau_2(x_1, y_1, x_2, y_2), \end{aligned}\right\} \tag{5}$$

reducing to $\alpha_0, \beta_0, \lambda_0, t_1, t_2$ for $(x_1, y_1, x_2, y_2) = (x_{10}, y_{10}, x_{20}, y_{20})$, and of class $C'$ near these values. Substituting from these last functions for $\alpha, \beta, \lambda$, in equations (5), we have a family of extremals

$$x = \phi(t, x_1, y_1, x_2, y_2), \quad y = \psi(t, x_1, y_1, x_2, y_2), \tag{6}$$

and two functions, $\tau_1(x_1, y_1, x_2, y_2)$ and $\tau_2(x_1, y_1, x_2, y_2)$ for which the following conditions are then satisfied:

$$\left.\begin{aligned} \phi(\tau_1, x_1, y_1, x_2, y_2) = x_1, \quad \phi(\tau_2, x_1, y_1, x_2, y_2) = x_2, \\ \psi(\tau_1, x_1, y_1, x_2, y_2) = y_1, \quad \psi(\tau_2, x_1, y_1, x_2, y_2) = y_2, \\ \int_{\tau_1}^{\tau_2} G(\phi, \psi, \phi', \psi')\, dt = k. \end{aligned}\right\} \tag{7}$$

By differentiating these we find that the following relations hold at the point $M_1$:

$$\left.\begin{aligned} \phi'\tau_{1x_1} + \phi_{x_1} = 1, \quad \psi'\tau_{1x_1} + \psi_{x_1} = 0, \\ \phi'\tau_{1y_1} + \phi_{y_1} = 0, \quad \psi'\tau_{1y_1} + \psi_{y_1} = 1, \\ \phi'\tau_{1x_2} + \phi_{x_2} = 0, \quad \psi'\tau_{1x_2} + \psi_{x_2} = 0, \\ \phi'\tau_{1y_2} + \phi_{y_2} = 0, \quad \psi'\tau_{1y_2} + \psi_{y_2} = 0, \end{aligned}\right\} \tag{8}$$

* Bolza, *loc. cit.*, p. 478.

while at the point $M_2$,

$$\left.\begin{aligned} \phi'\tau_{2x_1}+\phi_{x_1}=0, \quad & \psi'\tau_{2x_1}+\psi_{x_1}=0, \\ \phi'\tau_{2y_1}+\phi_{y_1}=0, \quad & \psi'\tau_{2y_1}+\psi_{y_1}=0, \\ \phi'\tau_{2x_2}+\phi_{x_2}=1, \quad & \psi'\tau_{2x_2}+\psi_{x_2}=0, \\ \phi'\tau_{2y_2}+\phi_{y_2}=0, \quad & \psi'\tau_{2y_2}+\psi_{y_2}=1. \end{aligned}\right\} \tag{9}$$

In accordance with the notation and nomenclature of Bolza,* we use the notation

$$I(x_1, y_1, x_2, y_2) = \int_{\tau_1}^{\tau_2} F(\phi, \psi, \phi', \psi')\, dt, \tag{10}$$

and call this expression the extremal integral. We desire to obtain the partial derivatives of the function $I$ with respect to its four arguments. In order to simplify the results we make use of two important functions $\Omega$ and $\Psi$, which will now be introduced.

The problem under consideration may be interpreted as a problem in space by defining a third coordinate $z$ by the equation

$$z=\chi(t, x_1, y_1, x_2, y_2) = \int_{\tau_1}^{t} G(\phi, \psi, \phi', \psi')\, dt, \tag{11}$$

where $\phi$, $\psi$ are of the form given in (6). By differentiation of (11) with respect to $t$ and $\alpha$, where $\alpha$ is an arbitrarily selected one of the elements $x_1, y_1, x_2, y_2$, we obtain

$$-\chi_t+G(\phi, \psi, \phi', \psi')=0, \quad \chi_a = \int_{\tau_1}^{t} (G_x\phi_a+G_y\psi_a+G_{x'}\phi'_a+G_{y'}\psi'_a) - G|^1\tau_{1a}. \tag{12}$$

If $h$ is defined by the equation

$$h=F+\lambda(G-z')=H-\lambda z',$$

then the equations

$$h_x-\frac{d}{dt}h_{x'}=0, \quad h_y-\frac{d}{dt}h_{y'}=0, \quad h_z-\frac{d}{dt}h_{z'}=0 \tag{13}$$

are satisfied along any extremal arc, and might be called the Euler equations of the space problem.

We now set up the function $\Omega$ by means of the following equation:

$$\begin{aligned} 2\Omega=&\begin{pmatrix} H_{xx} H_{xy} \\ H_{yx} H_{yy} \end{pmatrix}(\xi, \eta)\,(\xi, \eta)+2\begin{pmatrix} H_{x'x} H_{x'y} \\ H_{y'x} H_{y'y} \end{pmatrix}(\xi, \eta)\,(\xi', \eta') \\ &+\begin{pmatrix} H_{x'x'} H_{x'y'} \\ H_{y'x'} H_{y'y'} \end{pmatrix}(\xi', \eta')\,(\xi', \eta')+2\mu(G_x\xi+G_y\eta+G_{x'}\xi'+G_{y'}\eta'-\zeta'), \end{aligned} \tag{14}$$

in which the notation is explained by the equation

$$\begin{pmatrix} A_{11} A_{12} \\ A_{21} A_{22} \end{pmatrix}(x_1, y_1)\,(x_2, y_2) = (A_{11}x_1 + A_{12}y_1)x_2 + (A_{21}x_1 + A_{22}y_1)y_2.$$

* *Loc. cit.*, pp. 308 ff.

Then the equations analogous to those of Jacobi for other problems in the Calculus of Variations are the following:

$$\left.\begin{aligned}
\Omega_\xi-\frac{d}{dt}\Omega_{\xi'}\equiv{}&H_{xx}\xi+H_{xy}\eta+H_{xx'}\xi'+H_{xy'}\eta'+G_x\mu\\
&-\frac{d}{dt}(H_{x'x}\xi+H_{x'y}\eta+H_{x'x}\xi'+H_{x'y'}\eta'+G_{x'}\mu)=0,\\
\Omega_\eta-\frac{d}{dt}\Omega_{\eta'}\equiv{}&H_{yx}\xi+H_{yy}\eta+H_{yx'}\xi'+H_{yy'}\eta'+G_y\mu\\
&-\frac{d}{dt}(H_{y'x}\xi+H_{y'y}\eta+H_{y'x'}\xi'+H_{y'y'}\eta'+G_{y'}\mu)=0,\\
\Omega_\zeta-\frac{d}{dt}\Omega_{\zeta'}\equiv{}&\frac{d}{dt}\mu=0,\\
\Omega_\mu-\frac{d}{dt}\Omega_{\mu'}\equiv{}&G_x\xi+G_y\eta+G_{x'}\xi'+G_{y'}\eta'-\zeta'=0.
\end{aligned}\right\}\tag{15}$$

These equations are satisfied by the functions

$$(\xi,\eta,\zeta,\mu)=(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\tag{16}$$

obtained from (5), (6) and (12), where $\alpha$ stands for any one of $x_1, y_1, x_2, y_2$. This is proved by substituting the functions (6) in the Euler equations (1) and differentiating the resulting identities with respect to $\alpha$, and by differentiating (12) for $t$.

The expressions for the values of $\chi_\alpha$ at the points $M_1$ and $M_2$ will be useful in later simplifications, and will be computed now. The last equation of (7) is satisfied by the functions (6), and hence by differentiation we obtain

$$\int_{\tau_1}^{\tau_2}(G_x\phi_\alpha+G_y\psi_\alpha+G_{x'}\phi'_\alpha+G_{y'}\psi'_\alpha)\,dt+G|^2t_{1\alpha}-G|^1t_{1\alpha}=0,$$

that is, from (12),

$$\chi_\alpha(\tau_2,x_1,y_1,x_2,y_2)=-G|^1t_{1\alpha},\tag{17}$$

where it is to be remembered that $\alpha$ is now not one of the constants in (3), but one of the variables $x_1, y_1, x_2, y_2$. By direct computation from the expression (12) for $\chi_\alpha$, we obtain

$$\chi_\alpha(\tau_2,x_1,y_1,x_2,y_2)=-G|^1\tau_{1\alpha}.\tag{18}$$

Since $\Omega$ is a quadratic form in $\xi, \eta, \zeta, \mu, \xi', \eta', \zeta', \mu'$, we have the relations

$$\begin{aligned}
&\Sigma(\xi\,\Omega_\xi+\xi'\Omega_{\xi'})=2\Omega,\\
&\Sigma(\xi_1\Omega_{\xi_2}+\xi'_1\Omega_{\xi'_2})=\Sigma(\xi_2\Omega_{\xi_1}+\xi'_2\Omega_{\xi'_1}),
\end{aligned}$$

where $\Sigma$ denotes summation over the four elements $\xi, \eta, \zeta, \mu$, and the notation $\Omega_{\xi_2}$, for example, denotes the function obtained by differentiating $\Omega$ with

respect to $\xi$, and then replacing $\xi$ by $\xi_2$, $\eta$ by $\eta_2$, etc. It follows from the second of these that

$$\Sigma\left\{\xi_1\left(\Omega_{\xi_2}-\frac{d}{dt}\Omega_{\xi'_2}\right)-\xi_2\left(\Omega_{\xi_1}-\frac{d}{dt}\Omega_{\xi'_1}\right)\right\}=-\frac{d}{dt}\Psi\begin{pmatrix}\xi_1,\eta_1,\zeta_1,\mu_1,\xi'_1,\eta'_1,\zeta'_1,\mu'_1\\ \xi_2,\eta_2,\zeta_2,\mu_2,\xi'_2,\eta'_2,\zeta'_2,\mu'_2\end{pmatrix},$$

where

$$\Psi=\Sigma(\xi_1\Omega_{\xi'_2}-\xi_2\Omega_{\xi'_1}).$$

We see, therefore, that the function $\Psi$ is a constant if $(\xi_1, \eta_1, \zeta_1, \mu_1)$ and $(\xi_2, \eta_2, \zeta_2, \mu_2)$ are both solutions of equations (15).

We may now proceed to the computation of the derivatives of the extremal integral. Let $\alpha$, $\beta$ represent any two (or possibly both the same one) of $x_1, y_1, x_2, y_2$. By differentiating (10) we obtain

$$I_\alpha=\int_{\tau_1}^{\tau_2}(F_x\phi_\alpha+F_y\psi_\alpha+F_{x'}\phi'_\alpha+F_{y'}\psi'_\alpha)\,dt+F\tau_\alpha\Big|_1^2. \tag{19}$$

From the isoperimetric condition we have

$$0=\lambda\int_{\tau_1}^{\tau_2}(G_x\phi_\alpha+G_y\psi_\alpha+G_{x'}\phi'_\alpha+G_{y'}\psi'_\alpha)\,dt+\lambda G\tau_\alpha\Big|_1^2.$$

Then by adding, performing the Lagrangian partial differentiation, applying the Euler equation (1), and using the well-known homogeneity relation

$$H=x'H_{x'}+y'H_{y'},$$

we obtain

$$I_\alpha=H_{x'}(\phi'\tau_\alpha+\phi_\alpha)+H_{y'}(\psi'\tau_\alpha+\psi_\alpha)\Big|_1^2. \tag{20}$$

Similarly,

$$I_\beta=H_{x'}(\phi'\tau_\beta+\phi_\beta)+H_{y'}(\psi'\tau_\beta+\psi_\beta)\Big|_1^2. \tag{21}$$

Differentiating $I_\alpha$ with respect to $\beta$ we have

$$\begin{aligned}I_{\alpha\beta}=&(\phi'\tau_\alpha+\phi_\alpha)\left[\tau_\beta\frac{d}{dt}H_{x'}+\Omega_{\xi'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)\right]\\&+(\psi'\tau_\alpha+\psi_\alpha)\left[\tau_\beta\frac{d}{dt}H_{y'}+\Omega_{\eta'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)\right]\end{aligned} \tag{22}$$

since the values of $\phi'\tau_\alpha+\phi_\alpha$ and $\psi'\tau_\alpha+\psi_\alpha$ at the end-points are independent of $x_1, y_1, x_2, y_2$, in every case, as follows from equations (8) and (9). The arguments indicated in the derivatives of $\Omega$ are substituted for $\xi, \eta, \zeta, \mu$. Similarly, we obtain

$$\begin{aligned}I_{\beta\alpha}=&(\phi'\tau_\beta+\phi_\beta)\left[\tau_\alpha\frac{d}{dt}H_{x'}+\Omega_{\xi'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\right]\\&+(\psi'\tau_\beta+\psi_\beta)\left[\tau_\alpha\frac{d}{dt}H_{y'}+\Omega_{\eta'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\right]\Big|_1^2.\end{aligned} \tag{23}$$

To verify the computations, we prove the equality of these expressions for $I_{\alpha\beta}$ and $I_{\beta\alpha}$. By applying the Euler equations (1), and the relations

$$H_{xx'}\phi'+H_{xy'}\psi'=H_x, \qquad H_{yx'}\phi'+H_{yy'}\psi'=H_y,$$
$$H_{x'x'}\phi'+H_{x'y'}\psi'=0, \qquad H_{y'x'}\phi'+H_{y'y'}\psi'=0,$$
$$G_{x'}\phi'+G_{y'}\psi'=G,$$

all of which arise from the homogeneity conditions, we obtain the following:

$$\left.\begin{aligned} I_{\alpha\beta}&=\phi_\alpha\Omega_{\xi'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)+\psi_\alpha\Omega_{\eta'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)\\ &\quad+\tau_\alpha[H_x\phi_\beta+H_y\psi_\beta+G\lambda_\beta]+(\phi'\tau_\alpha+\phi_\alpha)\tau_\beta H_x+(\psi'\tau_\alpha+\psi_\alpha)\tau_\beta H_y|_1^2,\\ I_{\beta\alpha}&=\phi_\beta\Omega_{\xi'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)+\psi_\beta\Omega_{\eta'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\\ &\quad+\tau_\beta[H_x\phi_\alpha+H_y\psi_\alpha+G\lambda_\alpha]+(\phi'\tau_\beta+\phi_\beta)\tau_\alpha H_x+(\psi'\tau_\beta+\psi_\beta)\tau_\alpha H_y|_1^2. \end{aligned}\right\} \quad (24)$$

The notation $\Omega_{\xi'}$, $\Omega_{\eta'}$ is explained specifically in equations (15), reference to which shows that $\chi_\beta$, $\chi_\alpha$ do not actually occur. By the third equation of (15) together with (16), (17) and (18) we have

$$G\tau_\beta\lambda_\alpha|_1^2=\chi_\beta\Omega_{\xi'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)|_1^2,$$
$$G\tau_\alpha\lambda_\beta|_1^2=\chi_\alpha\Omega_{\xi'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)|_1^2.$$

If we make substitutions accordingly in the above expressions for $I_{\alpha\beta}$ and $I_{\beta\alpha}$, and form their difference, we have

$$\begin{aligned} I_{\alpha\beta}-I_{\beta\alpha}&=\phi_\alpha\Omega_{\xi'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)-\phi_\beta\Omega_{\xi'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\\ &\quad+\psi_\alpha\Omega_{\eta'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)-\psi_\beta\Omega_{\eta'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\\ &\quad+\chi_\alpha\Omega_{\xi'}(\phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta)-\chi_\beta\Omega_{\xi'}(\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha)\\ &=\Psi\begin{pmatrix}\phi_\alpha,\psi_\alpha,\chi_\alpha,\lambda_\alpha,\phi'_\alpha,\psi'_\alpha,\chi'_\alpha,\lambda'_\alpha\\ \phi_\beta,\psi_\beta,\chi_\beta,\lambda_\beta,\phi'_\beta,\psi'_\beta,\chi'_\beta,\lambda'_\beta\end{pmatrix}\Bigg|_1^2. \end{aligned}$$

But the two sets of arguments in $\Psi$ satisfy equations (15), and hence the function $\Psi$ in the last equation is independent of $t$. Its values at $t=\tau_1$ and $t=\tau_2$ are the same, and hence it follows that $I_{\alpha\beta}=I_{\beta\alpha}$.

The desired partial derivatives of the extremal integral may now be computed very easily. In the expression (20) for $I_\alpha$ let $\alpha$ take successively the values $x_1, y_1, x_2, y_2$. Making use of relations (8) and (9) we obtain

$$I_{x_1}=-H_{x'}|^1, \quad I_{y_1}=-H_{y'}|^1, \quad I_{x_2}=+H_{x'}|^2, \quad I_{y_2}=+H_{y'}|^2. \qquad (27)$$

Of the second derivatives ten are required, viz., $I_{x_1x_1}, I_{x_1y_1}, I_{y_1y_1}, I_{x_1y_2}, I_{y_1x_2}, I_{y_1y_2}, I_{x_1x_2}, I_{x_2x_2}, I_{x_2y_2}, I_{y_2y_2}$. With the exception of the cases in which the two subscripts are the same, the derivatives occur in two different forms, whose values, however, are of course the same. The following results are obtained most readily from (22) and (23). We employ relations (8) and (9) and the Euler equations (1).

$$
\begin{aligned}
&I_{x_1x_1}=-[H_x\tau_{x_1}+\Omega_{\xi'}(\phi_{x_1},\psi_{x_1},\chi_{x_1},\lambda_{x_1})]^1,\\
&I_{x_1y_1}=-[H_x\tau_{y_1}+\Omega_{\xi'}(\phi_{y_1},\psi_{y_1},\chi_{y_1},\lambda_{y_1})]^1=-[H_y\tau_{x_1}+\Omega_{\eta'}(\phi_{x_1},\psi_{x_1},\chi_{x_1},\lambda_{x_1})]^1,\\
&I_{y_1y_1}=-[H_y\tau_{y_1}+\Omega_{\eta'}(\phi_{y_1},\psi_{y_1},\chi_{y_1},\lambda_{y_1})]^1,\\
&I_{x_1x_2}=-[H_x\tau_{x_2}+\Omega_{\xi'}(\phi_{x_2},\psi_{x_2},\chi_{x_2},\lambda_{x_2})]^1=+[H_x\tau_{x_1}+\Omega_{\xi'}(\phi_{x_1},\psi_{x_1},\chi_{x_1},\lambda_{x_1})]^2,\\
&I_{x_1y_2}=-[H_x\tau_{y_2}+\Omega_{\xi'}(\phi_{y_2},\psi_{y_2},\chi_{y_2},\lambda_{y_2})]^1=+[H_y\tau_{x_1}+\Omega_{\eta'}(\phi_{x_1},\psi_{x_1},\chi_{x_1},\lambda_{x_1})]^2,\\
&I_{y_1x_2}=-[H_y\tau_{x_2}+\Omega_{\eta'}(\phi_{x_2},\psi_{x_2},\chi_{x_2},\lambda_{x_2})]^1=+[H_x\tau_{y_1}+\Omega_{\xi'}(\phi_{y_1},\psi_{y_1},\chi_{y_1},\lambda_{y_1})]^2,\\
&I_{y_1y_2}=-[H_y\tau_{y_2}+\Omega_{\eta'}(\phi_{y_2},\psi_{y_2},\chi_{y_2},\lambda_{y_2})]^1=+[H_y\tau_{y_1}+\Omega_{\eta'}(\phi_{y_1},\psi_{y_1},\chi_{y_1},\lambda_{y_1})]^2,\\
&I_{x_2x_2}=+[H_x\tau_{x_2}+\Omega_{\xi'}(\phi_{x_2},\psi_{x_2},\chi_{x_2},\lambda_{x_2})]^2,\\
&I_{x_2y_2}=+[H_x\tau_{y_2}+\Omega_{\xi'}(\phi_{y_2},\psi_{y_2},\chi_{y_2},\lambda_{y_2})]^2=+[H_y\tau_{x_2}+\Omega_{\eta'}(\phi_{x_2},\psi_{x_2},\chi_{x_2},\lambda_{x_2})]^2,\\
&I_{y_2y_2}=+[H_y\tau_{y_2}+\Omega_{\eta'}(\phi_{y_2},\psi_{y_2},\chi_{y_2},\lambda_{y_2})]^2.
\end{aligned}
$$

The derivatives $\Omega_{\xi'}$, $\Omega_{\eta'}$ are given by the formulas

$$
\begin{aligned}
\Omega_{\xi'}(\xi,\eta,\zeta,\mu)&=H_{x'x}\xi+H_{x'y}\eta+H_{x'x'}\xi'+H_{x'y'}\eta'+G_{x'}\mu,\\
\Omega_{\eta'}(\xi,\eta,\zeta,\mu)&=H_{y'x}\xi+H_{y'y}\eta+H_{y'x'}\xi'+H_{y'y'}\eta'+G_{y'}\mu.
\end{aligned}
$$

## § 3. *A New Necessary Condition.*

Suppose now that arc $E_{12}$ is an extremal satisfying the strengthened Legendre and Jacobi conditions and cut transversally by the arc $L$ at $P_1$ and $P_2$. Form the function

$$J(u,v)=\int_v^u F(\tilde{x},\tilde{y},\tilde{x}',\tilde{y}')\,dx+I(\tilde{x}(u),\tilde{y}(u),\tilde{x}(v),\tilde{y}(v)), \tag{29}$$

where the $\tilde{x}(x)$, $\tilde{y}(x)$ are those defining the arc $L$ given in Section 1. This function $J(u,v)$ must be a maximum at $(u,v)=(x_1,x_2)$ if arc $E_{12}$ is to maximize $J$ among arcs with end-points on $L$ which give to integral $K$ the value $k$. The functions $\alpha,\beta,\lambda,\tau_1,\tau_2$ in (5) become functions of $u$ and $v$ when $x_1,y_1,x_2,y_2$ are replaced by $\tilde{x}(u)$, $y(u)$, $\tilde{x}(v)$, $\tilde{y}(v)$, so that the family of extremals (3) or (6) has the form

$$x=\phi(t,u,v),\quad y=\psi(t,u,v). \tag{30}$$

Thus we have the relations

$$
\left.\begin{aligned}
&\phi(t,x_1,y_1,x_2,y_2)=\phi(t,u,v), \quad \psi(t,x_1,y_1,x_2,y_2)=\psi(t,u,v),\\
&\tau_1(x_1,y_1,x_2,y_2)\ =\tau_1(u,v), \quad \tau_2(x_1,y_1,x_2,y_2)\ =\tau_2(u,v),\\
&\lambda(x_1,y_1,x_2,y_2)\ =\lambda(u,v).
\end{aligned}\right\} \tag{31}
$$

Then by differentiation we obtain

$$
\left.\begin{aligned}
&\phi_u=\phi_{x_1}\tilde{x}_u+\phi_{y_1}\tilde{y}_u, \quad \phi_v=\phi_{x_2}x_v+\phi_{y_2}\tilde{y}_v,\\
&\psi_u=\psi_{x_1}\tilde{x}_u+\psi_{y_1}\tilde{y}_u, \quad \psi_v=\psi_{x_2}\tilde{x}_v+\psi_{y_2}\tilde{y}_v,\\
&\tau_{1u}=\tau_{1x_1}\tilde{x}_u+\tau_{1y_1}\tilde{y}_u, \quad \tau_{1v}=\tau_{1x_2}\tilde{x}_v+\tau_{1y_2}\tilde{y}_v,\\
&\tau_{2u}=\tau_{2x_1}\tilde{x}_u+\tau_{2y_1}\tilde{y}_u, \quad \tau_{2v}=\tau_{2x_2}\tilde{x}_v+\tau_{2y_2}\tilde{y}_v,\\
&\lambda_u=\lambda_{x_1}\tilde{x}_u+\lambda_{y_1}\tilde{y}_u, \quad \lambda_v=\lambda_{x_2}\tilde{x}_v+\lambda_{y_2}\tilde{y}_v.
\end{aligned}\right\} \tag{32}
$$

It is evident that

$$\begin{aligned}
\Omega_{\xi'}(\phi_u, \psi_u, \chi_u, \lambda_u) &= x_u\Omega_{\xi'}(\phi_{x_1}, \psi_{x_1}, \chi_{x_1}, \lambda_{x_1}) + \tilde{y}_u\Omega_{\xi'}(\phi_{y_1}, \psi_{y_1}, \chi_{y_1}, \lambda_{y_1}),\\
\Omega_{\eta'}(\phi_u, \psi_u, \chi_u, \lambda_u) &= \tilde{x}_u\Omega_{\eta'}(\phi_{x_1}, \psi_{x_1}, \chi_{x_1}, \lambda_{x_1}) + \tilde{y}_u\Omega_{\eta'}(\phi_{y_1}, \psi_{y_1}, \chi_{y_1}, \lambda_{y_1}),\\
\Omega_{\xi'}(\phi_v, \psi_v, \chi_v, \lambda_v) &= \tilde{x}_v\Omega_{\xi'}(\phi_{x_2}, \psi_{x_2}, \chi_{x_2}, \lambda_{x_2}) + \tilde{y}_v\Omega_{\xi'}(\phi_{y_2}, \psi_{y_2}, \chi_{y_2}, \lambda_{y_2}),\\
\Omega_{\eta'}(\phi_v, \psi_v, \chi_v, \lambda_v) &= \tilde{x}_v\Omega_{\eta'}(\phi_{x_2}, \psi_{x_2}, \chi_{x_2}, \lambda_{x_2}) + \tilde{y}_v\Omega_{\eta'}(\phi_{y_2}, \psi_{y_2}, \chi_{y_2}, \lambda_{y_2}).
\end{aligned}$$

From (29) by differentiating we obtain

$$\begin{aligned}
J_u(u,v) &= F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')|^1 + I_{x_1}\tilde{x}_u + I_{y_1}\tilde{y}_u,\\
J_v(u,v) &= -F(x, \tilde{y}, \tilde{x}', \tilde{y}')|^2 + I_{x_2}x_v + I_{y_2}\tilde{y}_v,\\
J_{uu}(u,v) &= \frac{d}{du}[F(x, y, \tilde{x}', \tilde{y}')|^1] + I_{x_1}\tilde{x}_{uu} + I_{y_1}\tilde{y}_{uu}\\
&\qquad + \tilde{x}_u(I_{x_1x_1}\tilde{x}_u + I_{x_1y_1}\tilde{y}_u) + \tilde{y}_u(I_{y_1x_1}\tilde{x}_u + I_{y_1y_1}\tilde{y}_u),\\
J_{uv}(u,v) &= \tilde{x}_u(I_{x_1x_2}\tilde{x}_v + I_{x_1y_2}\tilde{y}_v) + \tilde{y}_u(I_{y_1x_2}\tilde{x}_n + I_{y_1y_2}\tilde{y}_v),\\
J_{vv}(u,v) &= \frac{d}{dv}[-F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')|^2] + I_{x_2}\tilde{x}_{vv} + I_{y_2}\tilde{y}_{vv}\\
&\qquad + \tilde{x}_v(I_{x_2x_2}\tilde{x}_v + I_{x_2y_2}\tilde{y}_v) + \tilde{y}_v(I_{y_2x_2}\tilde{x}_v + I_{y_2y_2}\tilde{y}_v).
\end{aligned}$$

Substituting in these from (28), performing the indicated operations and making use of relations (32) and (33), we have the following:*

$$\left.\begin{aligned}
J_u(u,v) &= F(\tilde{x}, \tilde{y}, x', \tilde{y}') - \tilde{x}_uH_{x'} - \tilde{y}_uH_{y'}|^1,\\
J_v(u,v) &= -F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') + \tilde{x}_vH_{x'} + \tilde{y}_vH_{y'}|^2,\\
J_{uu}(u,v) &= \tilde{x}_{uu}A + \tilde{y}_{uu}B + F_x(\tilde{x}, \tilde{y}, x', \tilde{y}')\tilde{x}_u + F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{y}_u\\
&\quad - \tilde{x}_u\Omega_{\xi'_1} - \tilde{y}_u\Omega_{\eta'_1} - H_xt_u\tilde{x}_u - H_yt_u\tilde{y}_u|^1,\\
J_{uv}(u,v) &= -\tilde{x}_u\Omega_{\xi'_2} - \tilde{y}_u\Omega_{\eta'_2} - H_xt_vx_u - H_yt_vy_u|^1,\\
&= x_v\Omega_{\xi'_1} + \tilde{y}_v\Omega_{\eta'_1} + H_xt_u\tilde{x}_v + H_yt_uy_v|^2,\\
J_{vv}(u,v) &= -\tilde{x}_uA - \tilde{y}_{vv}B - F_x(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')x_v - F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{y}_v\\
&\quad + \tilde{x}_v\Omega_{\xi'_2} + \tilde{y}_v\Omega_{\eta'_2} + H_xt_v\tilde{x}_v + H_yt_vy_v|^2,
\end{aligned}\right\} \quad (34)$$

where

$$A = F_{x'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{x'}, \quad B = F_{y'}(x, \tilde{y}, \tilde{x}', \tilde{y}') - H_{y'}, \qquad (35)$$

and the subscript 1 of $\xi'$ and $\eta'$ denotes that the arguments of $\Omega_{\xi'}$, $\Omega_{\eta'}$ are $\phi_u, \psi_u, \phi'_u, \psi'_{u_1}, \lambda_u$, while the subscript 2 denotes that the arguments are $\phi_v$, $\psi_v, \phi'_v, \psi'_v, \lambda_v$.

Since arc $E_{12}$ is cut transversally by $L$, it follows from (2) that when $(u, v) = (x_1, x_2)$,

$$J_u(x_1, x_2) = J_v(x_1, x_2) = 0.$$

---

*When the arguments are omitted in $F$, $G$, $H$ and their derivatives, they are understood to be $\phi, \psi, \phi', \psi'$.

The former of these two equalities may be written in the form

$$[F_{x'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{x'}]\tilde{x}_u + [F_{y'}(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}') - H_{y'}]\tilde{y}_u|^1 = 0,$$

or

$$A\tilde{x}_u + B\tilde{y}_u|^1 = 0,$$

where $A$ and $B$ are the functions defined by (35). Then there exists a function $m_1$ such that at the point $P_1$ the following relations hold:

$$A|^1 = \frac{-m_1\tilde{y}_u}{(\sqrt{\tilde{x}_u^2 + \tilde{y}_u^2})^3}, \quad B|^1 = \frac{m_1\tilde{x}_u}{(\sqrt{x_u^2 + \tilde{y}_u^2})^3}. \tag{36}$$

By a similar argument we have at the point $P_2$ the relations

$$A|^2 = \frac{-m_2\tilde{y}_v}{(\sqrt{\tilde{x}_v^2 + \tilde{y}_v^2})^3}, \quad B|^2 = \frac{m_2\tilde{x}_v}{(\sqrt{\tilde{x}_v^2 + \tilde{y}_v^2})^3}. \tag{37}$$

By substitution from (36) and (37) it follows that $J_{uu}$, $J_{uv}$, $J_{vv}$ have the forms

$$J_{uu}(u, v) = \frac{m_1}{r_1} + R_1, \quad J_{uv}(u, v) = S_1 = -S_2, \quad J_{vv}(u, v) = -\frac{m_2}{r_2} - R_2,$$

where

$$m_1 = (A^2 + B^2)|^1\sqrt{\tilde{x}_u^2 + \tilde{y}_u^2}, \qquad m_2 = (A^2 + B^2)|^2\sqrt{\tilde{x}_v^2 + \tilde{y}_v^2}, \tag{38}$$

$$\left.\begin{aligned} R_1 = F_x(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{x}_u + F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{y}_u \\ -\tilde{x}_u\Omega_{\xi'_1} - \tilde{y}_u\Omega_{\eta'_1} - H_x t_u\tilde{x}_u - H_y t_u\tilde{y}_u|^1, \\ R_2 = F_x(\tilde{x}, \tilde{y}, x', \tilde{y}')\tilde{x}_v + F_y(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\tilde{y}_v \\ -\tilde{x}_v\Omega_{\xi'_2} - \tilde{y}_v\Omega_{\eta'_2} - H_x t_v\tilde{x}_v - H_y t_v\tilde{y}_v|^2, \end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} S_1 = -x_u\Omega_{\xi'_2} - \tilde{y}_u\Omega_{\eta'_2} - H_x t_v\tilde{x}_u - H_y t_v\tilde{y}_u|^1, \\ S_2 = -\tilde{x}_v\Omega_{\xi'_1} - \tilde{y}_v\Omega_{\eta'_2} - H_x t_u\tilde{x}_v - H_y t_u\tilde{y}_v|^2, \end{aligned}\right\} \tag{40}$$

and $r_1$ and $r_2$ are the radii of curvature of $L$ at the points $P_1$ and $P_2$, respectively.

If arc $E_{12}$ is to give to the function $J(u, v)$ a maximum value, it is necessary not only that all preceding conditions be satisfied, but also that the second partial derivatives of $J(u, v)$ satisfy the following conditions:

$$J_{uu}(u, v) \leqq 0, \quad J_{vv}(u, v) \leqq 0, \quad J_{uu}(u, v)J_{vv}(u, v) - J_{uv}^2(u, v) \geqq 0.$$

Thus we have a new condition for the problem, analogous to the Jacobi condition in other problems in the Calculus of Variations. We may summarize our results as follows:

*Assumed that the Euler, Weierstrass, Legendre and Jacobi conditions are satisfied for the corresponding problem with fixed end-points—the Legendre and Jacobi in the stronger form—and that the arc $E_{12}$ is cut transversally by*

*the fixed curve L, then a further necessary condition that arc E shall furnish a maximum for the problem at hand is that*

$$\left.\begin{aligned} &\text{(a)}\quad \frac{m_1}{r_1}+R_1\leqq 0, \quad -\frac{m_2}{r_2}-R_2\leqq 0,\\ &\text{(b)}\quad -\frac{m_1m_2}{r_1r_2}-\frac{m_1R_2}{r_1}-\frac{m_2R_1}{r_2}-R_1R_2+S_1S_2\geqq 0,\end{aligned}\right\} \tag{41}$$

where the notation is explained by equations (38), (39) and (40).

## § 4. *Sufficient Conditions.*

In the determination of conditions which are sufficient, direct application is made of a theorem proved by Hahn.* This theorem holds for the general Lagrangian problem, where certain isoperimetric conditions are to be fulfilled while the end-points of the comparison arcs are to satisfy any number of pre-assigned conditions. In so far as it relates to the problem at hand, this theorem may be formulated as follows:

Let $E_{12}$ be an extremal arc

$$x=\phi(t), \quad y=\psi(t), \quad t_1\leqq t\leqq t_2,$$

satisfying the Euler, Weierstrass, Legendre and Jacobi conditions,† the last three in the stronger forms. Then there exist weak neighborhoods $(E_{12})'_\rho$, $(E_{12})'_\sigma$, $\sigma\leqq\rho$, such that every extremal arc $E_{34}$ in $(E_{12})'_\sigma$ furnishes a maximum $I(E_{34})$ for the integral

$$I=\int_{t_1}^{t_2}F(x, y, x', y')\,dt$$

with respect to all arcs $V_{34}$ in $(E_{12})'_\rho$ such that $K(V_{34})=K(E_{34})$.

Let $E_{34}$ be an extremal arc joining a point $P_3(x=x_3)$ to a point $P_4(x=x_4)$ on $L$ and having $K(E_{34})=K(E_{12})$. The extremals $E_{34}$ form a two-parameter family with parameters $x_3$, $x_4$, containing $E_{12}$ for $x_3=x_1$, $x_4=x_2$. If the conditions given in the theorem of § 3 are changed by the substitution, in the place of (41), of the conditions

$$\left.\begin{aligned} &\text{(a)}\quad \frac{m_1}{r_1}-R_1<0, \quad -\frac{m_1m_2}{r_1r_2}-R_2<0,\\ &\text{(b)}\quad \frac{m_1m_2}{r_1r_2}-\frac{m_1R_2}{r_1}-\frac{m_2R_1}{r_2}-R_1R_2+S_1S_2>0,\end{aligned}\right\} \tag{41'}$$

* See Hahn, "Ueber Variations Probleme mit variablen Endpunkten," *Monatshefte für Mathematik und Physik*, Vol. XXII (1911), p. 127.

† These are conditions I, II', III', IV' of Bolza, *loc. cit.*, p. 514 with the proper changes for the determination of a maximum instead of a minimum.

we have a set of conditions which are sufficient to insure that $E_{12}$ furnishes a maximum among the curves $E_{34}$; *i. e.*, that

$$J(E_{12}) > J(E_{34}), \quad E_{34} \neq E_{12}.$$

Now if $\tau \leqq \sigma$ is sufficiently small, every arc $V_{34}$ with $K(V_{34}) = K(E_{12})$ in $(E_{12})'_\tau$ determines an extremal $E_{34}$ in $(E_{12})'_\sigma$ of the Hahn theorem with $K(E_{34}) = K(V_{34}) = K(E_{12})$, and so near $E_{12}$ that

$$J(E_{34}) < J(E_{12}),$$

according to the preceding paragraph. If then $E_{12}$ satisfies in addition to the above conditions the Weierstrass condition in the stronger form, the hypothesis of the Hahn theorem is satisfied and consequently we have

$$J(V_{34}) < J(E_{34}).$$

Hence it follows that

$$J(V_{34}) < J(E_{12})$$

for all variation arcs $V_{34}$ with $K(V_{34}) = K(E_{12})$ in $(E_{12})'_\tau$. We have thus proved the theorem:

*If $E_{12}$ satisfies the Euler condition, the transversality condition, and* (41'), *together with the stronger forms of the Weierstrass, Legendre and Jacobi conditions, then there exists a $\tau$ such that $E_{12}$ furnishes a maximum $J(E_{12})$ for the function*

$$J = \int_{L_{21}} F(\tilde{x}, \tilde{y}, \tilde{x}', \tilde{y}')\, dx + \int_{E_{12}} F(x, y, x', y') \cdot dt$$

*with respect to all arcs $V_{34}$ in $(E_{12})'_\tau$ with $K(V_{34}) = K(E_{12})$.*

## § 5. *Geometric Interpretations.*

The conditions found in Section 4 may be interpreted geometrically by the use of a set of oblique axes in the plane. The coordinates of points in the plane referred to this set of axes will be considered as possible values of radii of curvature of the curve $L$ at the points $P_1$ and $P_2$. Critical points are those at which the equalities

$$\left.\begin{aligned} &\text{(a)} \quad r_1 + \frac{m_1}{R_1} = 0, \quad r_2 + \frac{m_2}{R_2} = 0, \\ &\text{(b)} \quad r_1 r_2 + \frac{m_2 R_1 r_1}{R_1 R_2 + S_1^2} + \frac{m_1 R_2 r_2}{R_1 R_2 + S_1^2} + \frac{m_1 m_2}{R_1 R_2 + S_1^2} = 0, \end{aligned}\right\} \tag{42}$$

are satisfied, special discussion being required when either $r_1$ or $r_2$ vanishes. In any particular case the signs of the various functions are of course

determined. We limit this discussion of the general problem to the case for which

$$m_1>0, \quad R_1<0, \quad m_2<0, \quad R_2>0, \quad c=R_1R_2+S_1^2<0,$$

and will consider that in the given situation, only $r_1$ and $r_2$ may vary.

We suppose that in the given situation, the extremal arc $E_{12}$ (see Fig. 3) is cut transversally by the fixed curve $L$ at $P_1$ and $P_2$. Through these two points draw lines $X_1'X_2$ and $X_2'X_2$ parallel to $r_1$ and $r_2$ respectively, and intersecting* at $O$. We take these lines as a set of oblique axes, $OX_1$ and $OX_2$ being the positive directions. The first two equalities of (42) determine two straight lines $M'M$ and $N'N$ parallel to the axes. The last equality of (42) is

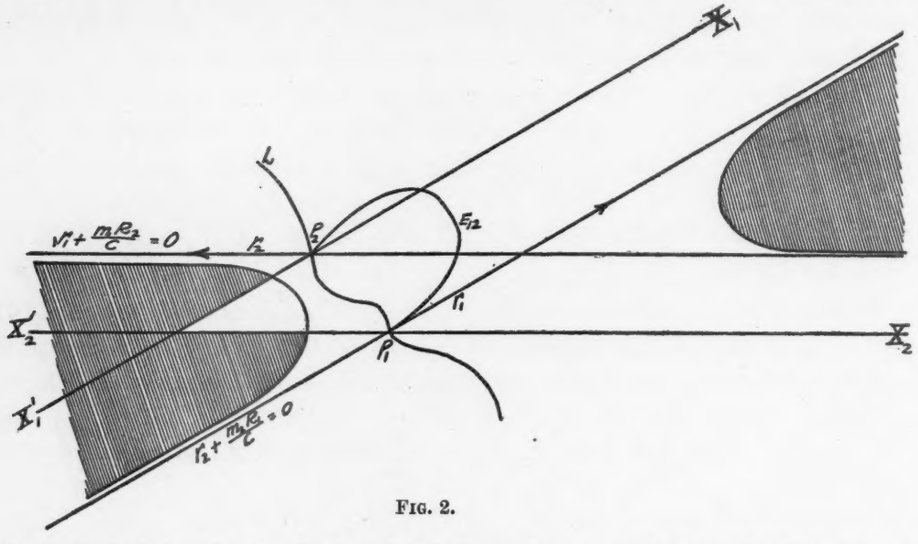


FIG. 2.

an hyperbola lying in the first and third quadrants and asymptotic to the two lines

$$r_1+\frac{m_1R_2}{c}=0, \quad r_2+\frac{m_2R_1}{c}=0.$$

Reference to condition (41) then shows that

*In order that two quantities $\rho_1, \rho_2$, be suitable values for $r_1$ and $r_2$ for the existence of a maximum, it is necessary and sufficient that the point $(\rho_1, \rho_2)$ lie in the portion of the plane shaded in Fig.* 2.

## § 6. *Application to the Problem of Dido.*

Let $L$ be a given fixed curve. We wish to determine an arc $E_{12}$ with endpoints $P_1$ and $P_2$ on $L$ and with a given length $k$, such that the area enclosed

*In case these lines do not intersect, some other set of lines may be used; for instance, line $r_1$ and the line perpendicular to it through $P$.

by the arc $P_2P_1$ of $L$ and the arc $P_1P_2$ of $E$ shall be a maximum, $P_1$ and $P_2$ to be distinct points.

The functions $F$ and $G$ for this problem are

$$F(x, y, x', y') = \tfrac{1}{2}(xy' - yx'), \quad G(x, y, x', y') = \sqrt{x'^2 + y'^2}.$$

Suppose that the equations of the fixed curve $L$ are

$$x = \tilde{x}(\varkappa), \quad y = \tilde{y}(\varkappa), \tag{L}$$

and denote by $u$ and $v$ the values of $\varkappa$ in the neighborhoods of the values of $\varkappa_1$ and $\varkappa_2$, which latter values define $P_1$ and $P_2$, respectively. Let the equation of $E_{12}$ be

$$x = \phi(t), \quad y = \psi(t), \quad t_1 \leqq t \leqq t_2. \tag{E}$$

We have then to maximize the function

$$J = \tfrac{1}{2}\int_u^v (\tilde{x}\tilde{y}' - \tilde{y}\tilde{x}')\, d\varkappa + \tfrac{1}{2}\int_{t_1}^{t_2} (\phi\psi' - \psi\phi')\, dt,$$

while the integral $K$:

$$K = \int_{t_1}^{t_2} \sqrt{\phi'^2 + \psi'^2}\, dt$$

is to have a preassigned value $k$.

Applying the results of §§ 1, 3, 4, we obtain the following conditions for the problem:

I. $E_{12}$ is the arc of a circle *

$$x = \alpha - \lambda \cos t, \quad y = \beta - \lambda \sin t, \quad t_1 \leqq t \leqq t_2.$$

II. $$E(x, y, p, q, x', y'; \lambda) = \lambda \frac{(px' - qy')^2}{\sqrt{p^2+q^2}\{\sqrt{p^2+q^2}\sqrt{x'^2+y'^2} + px' + qy'\}} \leqq 0,$$

*i. e.*, $\lambda \leqq 0$ for $(x, y, p, q)$ on $E_{12}$ for every $(x', y') \neq (0, 0)$. Since by condition I, $\lambda$ is seen to be the radius of the circular arc $E_{12}$, the value zero is necessarily excluded, and we have the stronger condition $\lambda < 0$.

III. $H_1 = \dfrac{\lambda}{(\sqrt{x'^2+y'^2})^3} \leqq 0$ along $E_{12}$. This condition follows directly from condition II, and in fact in the stronger form.

IV. $E_{12}$ contains no conjugate point to $P_1$ or $P_2$; that is, since $P_1$ and $P_2$ are distinct, $t_2 < t_1 + 2\pi$.

Now from I, $$\sqrt{\phi'^2 + \psi'^2} = -\lambda.$$

Since $E$ and $L$ intersect at $P_1$ we have

$$\phi'|^1 = \lambda \sin t_1, \quad \psi'|^1 = -\lambda \cos t_1.$$

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* As to the determination of conditions I–V, cf. also Bolza, *loc. cit.*, pp. 465, 483.

Accordingly we have for the transversality condition,

$$\tilde{x}'\sin t_1 - \tilde{y}'\cos t_1 = 0$$

and a similar result at $P_2$. We therefore have the condition

V. $E_{12}$ cuts the curve $L$ orthogonally at both $P_1$ and $P_2$.

If now the length of arc is chosen as the parameter $s$, we have from condition V the important relations:

$$\tilde{x}_u(u) = \cos t_1, \quad \tilde{y}_u(u) = \sin t_1,$$
$$\tilde{x}_v(v) = -\cos t_2, \quad \tilde{y}_v(v) = -\sin t_2.$$

Employing these relations and the values of $\alpha_u, \beta_u, \alpha_v, \beta_v$ from the Euler equations, we obtain from the last three expressions of (34) the following expressions for the second partial derivatives:

$$J_{uu}(u,v) = -\frac{\lambda}{r_1} - \frac{D_2}{D_1}, \quad J_{uv}(u,v) = \frac{D_3}{D_1}, \quad J_{vv}(u,v) = -\frac{\lambda}{r_2} - \frac{D_2}{D_1},$$

where

$$D_1 = 2 - 2\cos(t_2 - t_1) - (t_2 - t_1)\sin(t_2 - t_1),$$
$$D_2 = -\sin(t_2 - t_1) + (t_2 - t_1)\cos(t_2 - t_1),$$
$$D_3 = (t_2 - t_1) - \sin(t_2 - t_1),$$
$$r_1 = \frac{1}{|\tilde{x}_u\tilde{y}_{uu} - \tilde{y}_u\tilde{x}_{uu}|^1}, \quad r_2 = \frac{1}{|x_v\tilde{y}_{vv} - \tilde{y}_v\tilde{x}_{vv}|^2},$$

$r_1$ and $r_2$ being the radii of curvature of the curve $L$ at $P_1$ and $P_2$, respectively. Conditions (41) are therefore*

$$-\frac{\lambda}{r_1} - \frac{D_2}{D_1} \leqq 0, \quad -\frac{\lambda}{r_2} - \frac{D_2}{D_1} \leqq 0, \quad \left[-\frac{\lambda}{r_1} - \frac{D_2}{D_1}\right]\cdot\left[-\frac{\lambda}{r_2} - \frac{D_2}{D_1}\right] - \frac{D_3^2}{D_1^2} \geqq 0.$$

It is evident that if the first and third of these conditions are satisfied, the second must also hold.

We may state the result as follows:

*In order that $E_{12}$ and $L_{21}$ enclose a maximum area in the Problem of Dido stated above, it is necessary that $E_{12}$ be a circle-arc*

$$x = \alpha - \lambda\cos t, \quad y = \beta - \lambda\sin t, \quad t_1 \leqq t \leqq t_2,$$

*with $\lambda < 0$ and $t_2 < t_1 + 2\pi$, cutting $L$ orthogonally at $P_1$ and $P_2$, and that the conditions*

$$-\frac{\lambda}{r_1} - \frac{D_2}{D_1} \leqq 0, \quad \left(-\frac{\lambda}{r_1} - \frac{D_2}{D_1}\right)\left(-\frac{\lambda}{r_2} - \frac{D_2}{D_1}\right) - \frac{D_3^2}{D_1^2} \geqq 0$$

*be fulfilled.*

Applying the results of § 4, we have immediately the following:

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*With the first two of these conditions cf. Bolza, *loc. cit.*, p. 538, ex. 29.

*If $E_{12}$ is a circle-arc*

$$x=\alpha-\lambda\cos t,\quad y=\beta-\lambda\sin t,\quad t_1\leqq t\leqq t_2,$$

*with $\lambda<0$ and $t_2<t_1+2\pi$, cutting $L$ orthogonally at $P_1$ and $P_2$, and if in addition the conditions*

$$-\frac{\lambda}{r_1}-\frac{D_2}{D_1}<0,\quad \left(-\frac{\lambda}{r_1}-\frac{D_2}{D_1}\right)\left(-\frac{\lambda}{r_2}-\frac{D_2}{D_1}\right)-\frac{D_3^2}{D_1^2}>0$$

*be fulfilled, then there exists a $\tau$ such that the area enclosed by arcs $L_{21}$ and $E_{12}$ is the greatest among the areas enclosed by arcs $L_{43}$ and $V_{34}$, for all arcs $V_{34}$ in $(E_{12})'_\tau$ and of the same length as $E_{12}$.*

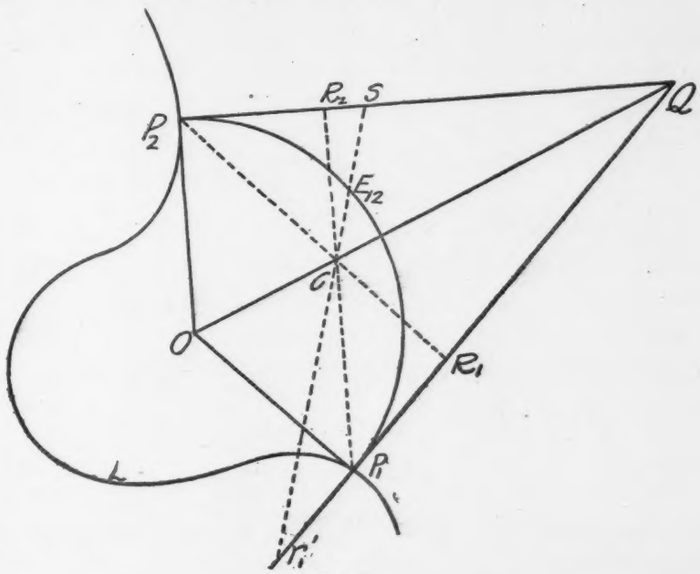


FIG. 3.

The geometric interpretation of § 5 is of course applicable to the Problem of Dido, but for this particular case, the following is a more direct interpretation. Suppose $L$ and $E_{12}$ are as in Fig. 3. $P_1Q$ and $P_2Q$ are tangents to $E_{12}$. Also

$$\angle P_1OQ=QOP_2=\omega=\frac{t_2-t_1}{2}.$$

If we measure $r_1$ from $P_1$ along $P_1Q$, and $r_2$ from $P_2$ along $P_2Q$, positive directions being $P_1$ to $Q$ and $P_2$ to $Q$, then the relation

$$\left(\frac{\lambda}{r_1}+\frac{D_2}{D_1}\right)\left(\frac{\lambda}{r_2}+\frac{D_2}{D_1}\right)-\frac{D_3^2}{D_1^2}=0 \tag{43}$$

relates projectively the points of the line $P_1Q$ to those of $P_2Q$. This pro-

jectivity is in fact a perspectivity, the center $C$ of the perspectivity being a point on $OQ$ at the distance $\frac{|\lambda|\sin\omega}{\omega}$ from $O$.

Let $R_1$ and $R_2$ be the points on $P_1Q$ and $P_2Q$ determined by $P_2C$ and $P_1C$, respectively. Then the condition

$$-\frac{\lambda}{r_1}-\frac{D_2}{D_1}<0$$

means that $r_1$ is not on the segment $P_1R_1$. Similarly, the condition

$$-\frac{\lambda}{r_2}-\frac{D_2}{D_1}<0$$

being fulfilled means that $r_2$ does not lie on $P_2R_2$.

Suppose now that a particular value $r_1'$ of $r_1$ be given, determining the point $r_1'$ in the figure. Draw $r_1'C$ cutting $P_2Q$ at $S$. Then the condition

$$\left(\frac{\lambda}{r_1}+\frac{D_2}{D_1}\right)\left(\frac{\lambda}{r_2}+\frac{D_2}{D_1}\right)-\frac{D_3^2}{D_1^2}>0$$

means that $r_2'$ may not lie on the segment $P_2S$.

It remains to prove the statement made above, that (43) relates perspectively the points of $P_1Q$ to those of $P_2Q$. In the figure take $OX$ as the positive $x$-axis, $O$ being the origin, of a set of perpendicular axes. Then $P_1$ is the point $(|\lambda|\cos\omega, -|\lambda|\sin\omega)$, while $P_2$ is $(|\lambda|\cos\omega, |\lambda|\sin\omega)$ and $C$ is $\left(\frac{|\lambda|\sin\omega}{\omega}, 0\right)$. The point on $P_1Q$ at the distance $r_1$ from $P_1$ is

$$(|\lambda|\cos\omega+r_1\sin\omega,\ -|\lambda|\sin\omega+r_1\cos\omega),$$

and the point on $P_2Q$ at the distance $r_2$ from $P_2$ is

$$(|\lambda|\cos\omega+r_2\sin\omega,\ |\lambda|\sin\omega-r_2\cos\omega).$$

The condition that these two latter points be collinear with $C$ is

$$\begin{vmatrix} |\lambda|\cos\omega+r_1\sin\omega & |\lambda|\cos\omega+r_2\sin\omega & \frac{|\lambda|\sin\omega}{\omega} \\ -|\lambda|\sin\omega-r\cos\omega & |\lambda|\sin\omega-r_2\cos\omega & 0 \\ 1 & 1 & 1 \end{vmatrix}=0.$$

After expansion and reduction this condition is found to be equivalent to relation (43).

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